MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J Fall 2018

Problem Set 6

Readings:
(a) Notes from Lecture 10 and 11.
(b) [Grimmett-Stirzaker]: Section 4.1-4.10. Optionally, Section 4.11.

Exercise 1. The probabilistic method. Twelve per cent of the circumference of a circle is colored blue, the rest is red. Show that, irrespective of the manner in which the colors are distributed, it is possible to inscribe a regular octagon in the circle with all its vertices red.

Hint: The probabilistic method is a general method for proving existence: if you can prove that a randomly selected structure has certain desired properties with some positive probability (no matter how small), then a structure with these properties is guaranteed to exist.

Exercise 2. Suppose X is a continuous random variable with a power law distribution. Namely there exists $c > 0$ and $\boldsymbol{\alpha} > 0$ such that $\mathbb{P}(X > x) = \frac{c}{x}$, for every $x \geq c$. Consider the r-th moment of X, namely $\mathbb{E}[X^r]$, where $r > 0$ is any real value. Find necessary and sufficient conditions for r in terms of c and for the r-th moment to be finite.

Exercise 3. We have a stick of unit length $[0, 1]$, and break it at X, where X is uniformly distributed on $[0, 1]$. Given the value x of X, we let Y be uniformly distributed on $[0, x]$, and let Z be uniformly distributed on $[0, 1-x]$. We assume that conditioned on $X = x$, the random variables Y and Z are independent. Find the joint PDF of Y and Z. Find $\mathbb{E}[X|Y]$, $\mathbb{E}[X|Z]$, and $\rho(Y, Z)$.

Exercise 4. Assume that $X_1, \ldots, X_n$ are independent continuous random variables with common density function function f. Let $X^{(1)}, \ldots, X^{(n)}$ be the ordered statistics of $X_1, \ldots, X_n$. Namely, $X^{(1)}$ is the smallest of $X_1, \ldots, X_n$, $X^{(2)}$ is the second smallest, etc., and $X^{(n)}$ is the largest of them all. Establish that the joint distribution of $X^{(1)}, \ldots, X^{(n)}$ is given by the joint density

$$f_{X^{(1)},\ldots,X^{(n)}}(x_1, \ldots, x_n) = n! f(x_1) \cdots f(x_n), \qquad x_1 < x_2 < \cdots < x_n,$$

and $f_{X^{(1)},\ldots,X^{(n)}}(x_1, \ldots, x_n) = 0$, otherwise. Use this to derive the densities for $\max_j X_j$ and $\min_j X_j$.

Exercise 5. Let $X_1, \dots, X_n$ be independent r.v. with $\mathrm{Exp}(\lambda)$ distribution. Consider $S_n = \sum_{1 \le j \le n} X_j$. The distribution of S_n is sometimes called *Erlang*.

(a) Establish that the density of S_n is $f_{S_n}(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} \exp(-\lambda x)$.
(A Gamma distribution with an integer shape parameter n.)

(b) Consider the joint distribution of $S_1, S_2, \dots, S_{n-1}$ given $S_n = x$. Establish that this joint distribution is the same as the joint distribution of $U^{(1)}, \dots, U^{(n-1)}$, where $U^{(1)}, \dots, U^{(n-1)}$ is the order statistics of $n-1$ independent r.v. with $U(0, x)$ distribution.

Exercise 6. A needle of length $2s < 1$ unit is randomly tossed onto a quad-ruled sheet with horizontal and vertical lines spaced at 1 unit. Assuming the position and the angle of the needle are independent and uniform, find the average number of lines the needle intersects.