MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J Fall 2018

Final Exam

Boilerplate:

- No collaboration
- No internet
- Closed books, Closed notes.
- Two (2-sided) cheat sheets allowed.
- Total: 100 pts
- **Partial credit will be given** (but please write clearly).

Exercise 1 (10 pts)**.** (Cautious Gambler's ruin) A gambler starts with $k \in [0, n]$ dollars and at each time step either skips a turn, or bets and wins 1 dollar, or bets and loses 1 dollar – all three cases happening with equal probability, independently across time. If he gets to n dollars, he stops and we say he "won". If he gets to 0 dollars he also stops and we say he is "ruined".

1. Show that eventually he must win or be ruined.
2. Find the probability that he wins.

Exercise 2 (20 pts)**.** Suppose $A_1, A_2, \ldots$ are independent events with $\mu_n = \sum\limits_{i=1}^{n} \mathbb{P}(A_i) \to \infty$ as $n \to \infty$. Let

$$X_n := \frac{1}{\mu_n} \sum_{i=1}^{n} 1_{A_i}$$

1. Prove $X_n \stackrel{\text{i.p.}}{\to} 1$
2. Prove $X_n \stackrel{L_1}{\to} 1$. (Hint: $\mathbb{E}[|V|] \leq \sqrt{\mathbb{E}[V^2]}$. What do you know about the variance of the sum of independent RVs?)
3. Prove $X_n \stackrel{\text{a.s.}}{\to} 1$ (Hint: first, let subsequence n_k be such that $(k-1)^4 \leq \mu_{n_k} \leq k^4$. What can you say about $\{|X_{n_k} - 1| > \frac{1}{k}\text{–i.o.}\}$?)

Exercise 3 (15 pts)**.** Two players are playing the following game. At time $t \geq 1$ both players generate random moves: player A's move is A_t and player B's move is B_t. The moves are iid and independent of each other. If $\sum_{s=1}^{t} A_s \geq \sum_{s=1}^{t} B_s$ then player A is declared a winner at step t (and we set $X_t = 0$), otherwise we say player B is a winner at step t (and set $X_t = 1$).

1. Let A_t be ± 2 with equal probability and B_t be ± 1 with equal probability. Find $\lim_{t\to\infty} \mathbb{P}[X_t = 1]$.
2. Now suppose both A_t and B_t are ± 1 with equal probability (and independently of each other). Find $\lim_{t\to\infty} \mathbb{P}[X_t = 1]$.
3. In the setting from part 2 show that X_t almost surely does not converge. (Hint: How many times do two symmetric random walks on $\mathbb{Z}$ meet?)

Exercise 4 (10pts)**.** Consider a finite-state homogeneous Markov chain with transition matrix $P(i,j)$. Suppose that for a certain state T we have $P(T,T) = 1$ and $P(i,T) = \epsilon > 0$ for all $i \neq T$. Let $X_0, X_1, \dots, X_n$ be a trajectory of this Markov chain, started from some distribution $X_0 \sim \pi$ with $\pi(T) < 1$. Show that conditioned on $X_n \neq T$ the law of the sequence $X_0, \dots, X_n$ is still a Markov chain. Find its transition matrix $\tilde{P}$ and initial distribution $\tilde{\pi}$ (i.e. $\tilde{\pi}$ is the law of X_0 given $\{X_n \neq T\}$). (Hint: you need to compute $\mathbb{P}[X_0 = a_0, \dots, X_n = a_n | X_n \neq T]$ and factorize it.)

Exercise 5 (10 pts)**.** Two friends are observing iid sequence $X_i \sim \mathrm{Ber}(p)$ with unknown $p \in [0,1]$, which they are trying to learn from the observations. They decide to record their observations as a running sum $S_t = \sum_{i=1}^t X_i$. Having observed n samples they start arguing. One says that they can write down the value S_n and forget $S_1, S_2, \dots, S_{n-1}$, since it won't help in determining p. The other one argues that there might be some useful information in the trajectory $S_1, \dots, S_{n-1}$ that will help learn p better. Who is right? (Hint: find $\mathbb{P}[S_1 = a_1, \dots S_{n-1} = a_{n-1} | S_n = a_n]$ as a function of p.)

Exercise 6 (15pts)**.** Let X_i be independent with $\mathbb{P}[X_i = \frac{1}{p_i}] = 1 - \mathbb{P}[X_i = 0] = p_i$. Let $M_t = \prod_{i=1}^t X_i$, and $M_0 = 1$. Denote $a = \prod_{i=1}^\infty p_i$.

1. Find $\mathbb{E}[M_t]$.
2. Show that M_t converges almost surely to a random variable M_∞ and find its distribution. (Hint: you may want to consider cases of $a = 0$ and $a > 0$ separately).
3. If $a = 0$ is collection $\{M_t, t = 0, 1, \dots\}$ uniformly integrable? (Hint: compute $\mathbb{E}[M_\infty]$)
4. If $a > 0$ is collection $\{M_t, t = 0, 1, \dots\}$ uniformly integrable?

Exercise 7 (20 pts)**.** Two drunks walk along a street with n blocks (and $n+1$ intersections labeled $0, 1, \dots, n$), starting at locations a and b (where a, b have the same parity, i.e. $a = b \bmod 2$). Two bars are located at 0 and n. Before he reaches a bar, each drunk performs a random walk, moving either left or right at every step with probability $1/2$ (independent of past steps); when a drunk arrives at a bar, he stops walking and goes in. If the two drunks meet they keep moving together ever after (still with probability 1/2 left-right).

What is the probability that the two drunks meet (either at a bar or while walking)? Consider three cases:

1. Their random walks are independent.
2. At each step they either both move right or both move left with equal probability. (If one of them is already captured by a bar, then only the remaining one keeps moving randomly.)
3. At each step they either both towards each other, or away from each other with equal probability. (If one of them is already captured by a bar, then only the remaining one keeps moving randomly.)

Hint: Think about union and intersection of events "left drunk goes to right bar", "right drunk goes to left bar".

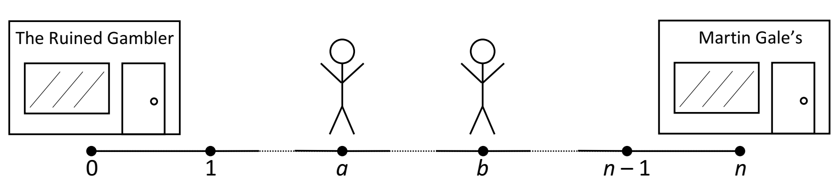

Figure 1: The drunks at the start of the process.