A Coupling Example

Recall the Ehrenfest Chain $\{X_t\}$ – we have n particles, each in one of two boxes (left or right). The *state* of the chain is simply the number of particles currently in the left box (and therefore ranges from 0 to n). At every step, a particle is chosen uniformly at random (independent of the past) and then moved to the other pox. Therefore, the transition kernel is

$$P(i,j) := \begin{cases} \frac{i}{n} & \text{if } j = i-1 \\ \frac{n-i}{n} & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

We define $\mathbb{E}^i[\tau_j]$ to be the expected number of steps to reach state j starting at state i.

Problem 0.1. Use *coupling* to show that $\mathbb{E}^0[\tau_{n/3}]$ is linear in n (so $X_0 = 0$).

Remember that *coupling* is running two stochastic processes off the same randomness in some way, so as to get the desired result. Note that so long as $t \le \tau_{n/3}$ (where $X_0 = 0$) we have $X_t \le n/3$ and therefore

$$\mathbb{P}[X_{t+1} = X_t + 1] \ge 2/3 \text{ and } \mathbb{P}[X_{t+1} = X_t - 1] \le 1/3$$

Therefore, we'll couple this to a weighted random walk $\{Y_t\}$ (with $Y_0 = 0$) with probability 2/3 to increment every step, and 1/3 to decrement. Formally, we do this coupling by defining $U_t \overset{iid}{\sim} \text{unif}[0,1]$ and then defining for $t \ge 1$:

$$X_t = X_{t-1} + \mathbf{1}_{U_t \ge 1 - \frac{X_t}{n}} - \mathbf{1}_{U_t < \frac{X_t}{n}} \text{ and } Y_t = Y_{t-1} + \mathbf{1}_{U_t \ge 2/3} - \mathbf{1}_{U_t < 1/3}$$

It's easy to confirm that $\{X_t\}$ and $\{Y_t\}$ behave (marginally) as required; it's also easy to confirm (by induction) that for all $t \le \tau_{n/3}$ we have $Y_t \le X_t$, because $X_0 = Y_0 = 0$, and $X_t - X_{t-1} \ge Y_t - Y_{t-1}$ for such t. Therefore, defining $\tau^*_{n/3} := \min_t\{Y_t \ge n/3\}$, we have $\tau^*_{n/3} \ge \tau_{n/3}$; furthermore, $\{Y_t\}$ is much easier to analyze than $\{X_t\}$. Therefore $\mathbb{E}^0[\tau^*_{n/3}] \ge \mathbb{E}^0[\tau_{n/3}]$.

(To intuitively see what $\tau^*_{n/3} \ge \tau_{n/3}$, imagine two runners running a race – if we know that runner 1 is always faster than runner 2, then runner 1 will finish faster than runner 2; and no information about exactly how fast they're running at any particular time is necessary to conclude this.)

Now we can analyze $\mathbb{E}^0[\tau^*_{n/3}]$. Let us define $a_k := \mathbb{E}^{n/3-k}[\tau^*_{n/3}]$. Obviously, $a_0 = 0$; and by the recursion rules we learned give that for $k \ge 1$,

$$a_k = \frac{1}{3}a_{k+1} + \frac{2}{3}a_{k-1} + 1$$

This is solved by $a_k = 3k$. Therefore, $\mathbb{E}^0[\tau^*_{n/3}] = a_{n/3} = n$. Therefore, $\mathbb{E}^0[\tau_{n/3}] \le n$, so we have a linear upper bound. But also we have a trivial linear lower bound because at least $n/3$ steps are necessary for the Ehrenfest chain to make it from 0 to $n/3$. Therefore

$$n/3 \le \mathbb{E}^0[\tau_{n/3}] \le n \implies \mathbb{E}^0[\tau_{n/3}] = \Theta(n)$$