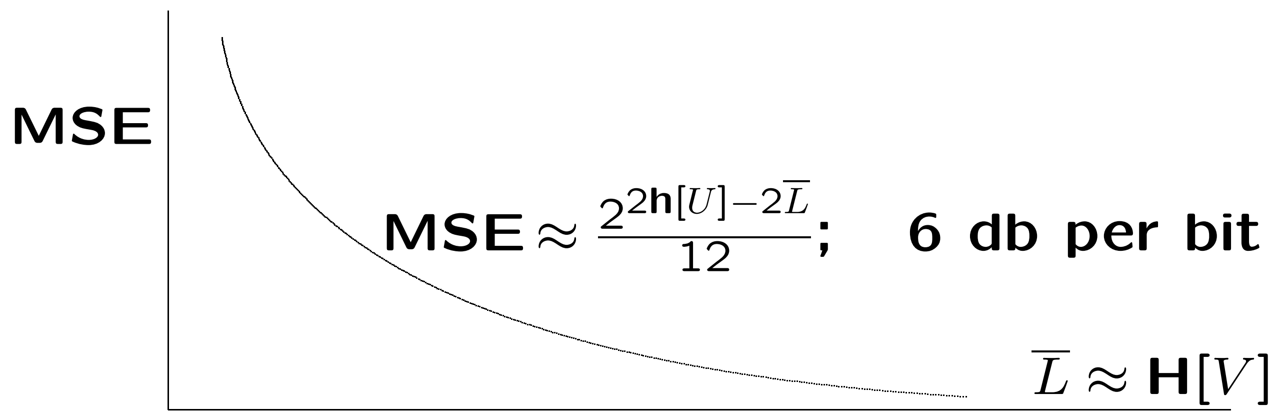
**Summary: For a high-rate, uniform scalar quantizer with interval $\Delta$,**

$$\begin{aligned}
\mathbf{h}(U) &= \int -f(u)\log[f(u)]\,du \approx \sum_j -\Delta f(j\Delta)\log[f(j\Delta)] \\
&\approx \sum_j -p_j \log[\frac{p_j}{\Delta}] = \mathbf{H}(V) + \log\Delta
\end{aligned}$$

$$\mathbf{H}(V) \approx \mathbf{h}(U) - \log\Delta; \qquad \textbf{MSE} \approx \frac{\Delta^2}{12}$$

$\mathbf{h}(U) - \log\Delta$ **is invariant to scaling.**



**As $\Delta \to 0$, i.e., H$(V) \to \infty$, the uniform quantizer approaches optimality.**

**For vector quantization, uniform quantization again approaches optimal for memoryless source.**

**Here a shaping gain is possible (replace square regions by hexagonal regions in 2D case).**

**The gain is not impressive; the MSE decreases by a factor of 1.039.**

# ANALOG SOURCE TO BIT STREAM

Why?

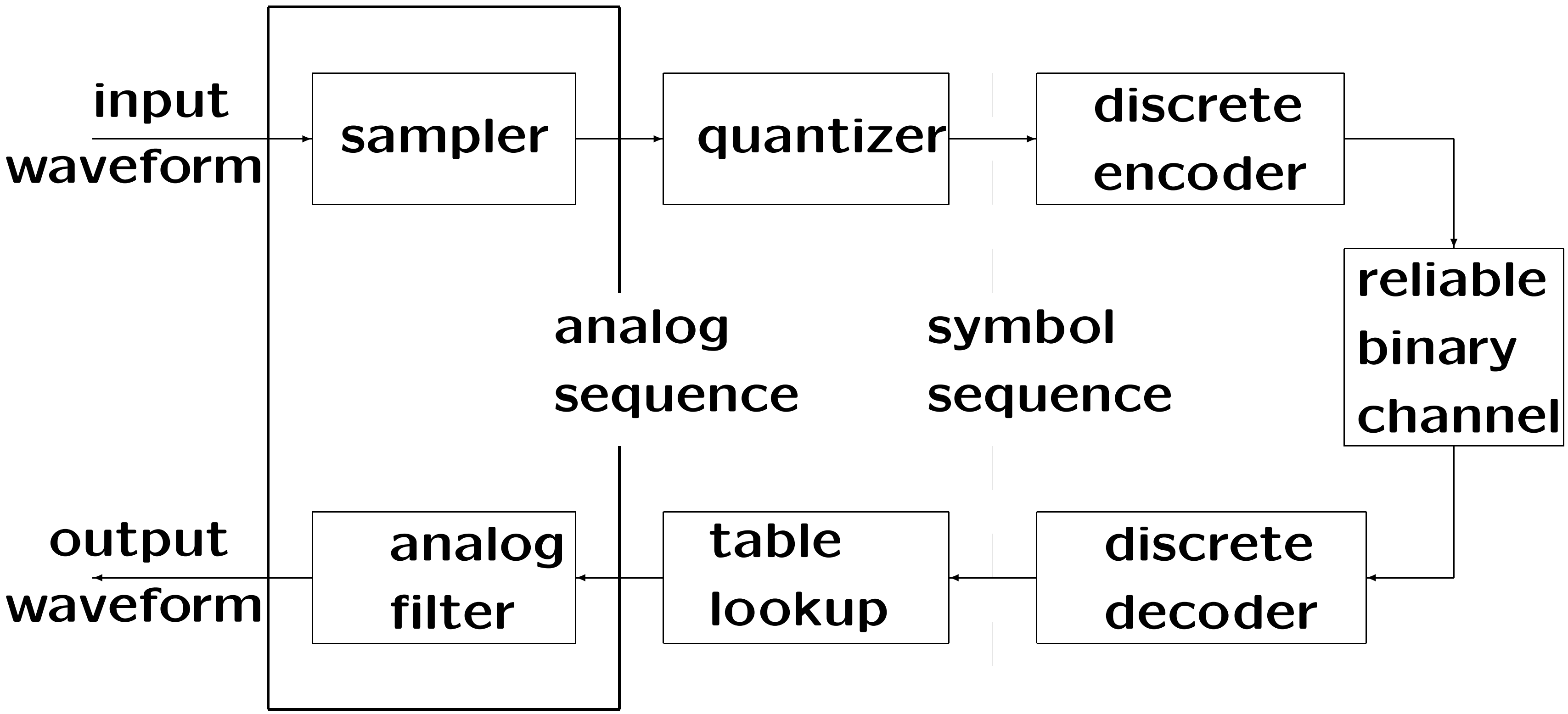
- Standard Binary interface separates source and channel coding

- Multiplex data on high speed channels.

- Digital data can be "cleaned up" at each link in a network.

- Can separate problems of waveform sampling from quantization from discrete source coding.

# WAVEFORM → SEQUENCE




Sampling is only one way to go from waveform to sequence.

Filtering is only one way to go back.

# FOURIER SERIES

**The Fourier series of a time-limited function maps function to a sequence of coefficients.**

**Let $u(t) = 0$ for $t < -T/2$ and $t > T/2$. Then**

$$u(t) = \begin{cases} \sum_{k=-\infty}^{\infty} \hat{u}_k \, e^{2\pi ikt/T} & \textbf{for } -T/2 \le t \le T/2 \\ 0 & \textbf{elsewhere,} \end{cases}$$

**where the complex coefficients $\hat{u}_k$ satisfy**

$$\hat{u}_k = \frac{1}{T} \int_{-T/2}^{T/2} u(t) e^{-2\pi ikt/T} \, dt, \qquad -\infty < k < \infty.$$

**This works for $u(t)$ complex as well as real.**

**To verify the formula for $\widehat{u}_k$,**

$$\begin{aligned}\int_{-T/2}^{T/2} u(t)e^{-2\pi ikt/T}\,dt &= \int_{-T/2}^{T/2} \sum_m \widehat{u}_m e^{2\pi i(m-k)t/T}\,dt \\ &= \widehat{u}_k \int_{-T/2}^{T/2} dt \;=\; T\widehat{u}_k.\end{aligned}$$

**Repeating the same kind of argument,**

$$\int_{-T/2}^{T/2} |u(t)|^2\,dt = T\sum_k |u_k|^2$$

**If we represent $u(t)$ by $\{\widehat{u}_k; k \in \mathbb{Z}\}$, then quantize each $\widehat{u}_k$ to $\widehat{v}_k$ and reconstruct $v(t)$,**

$$\int_{-T/2}^{T/2} |u(t)-v(t)|^2\,dt \;=\; T\sum_k |\widehat{u}_k - \widehat{v}_k|^2$$

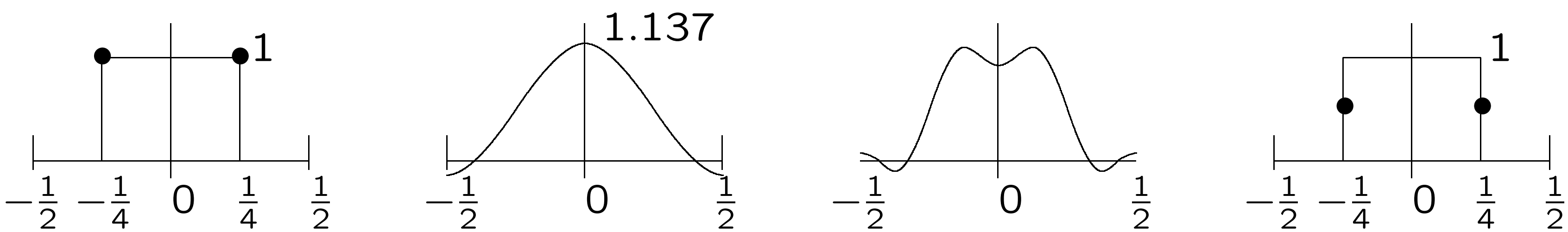
**Define the standard rectangular function**

$$\text{rect}(t) = \begin{cases} 1 & \textbf{for } -1/2 \leq t \leq 1/2 \\ 0 & \textbf{elsewhere,} \end{cases}$$

**Then** $u(t)$ **can be expressed as**

$$u(t) = \sum_{k=-\infty}^{\infty} \hat{u}_k \, e^{2\pi i k t/T} \, \text{rect}(\frac{t}{T}).$$

**Example: Suppose we expand the function** rect$(t/2)$ **in a Fourier series over** $[-\frac{1}{2}, \frac{1}{2}]$**.**




$u(t) = \text{rect}(2t)$ $\quad$ $\frac{1}{2} + \frac{2}{\pi}\cos(2\pi t)$ $\quad$ $\frac{1}{2} + \frac{2}{\pi}\cos(2\pi t) - \frac{2}{3\pi}\cos(6\pi t)$ $\quad$ $\sum_k u_k e^{2\pi i k t}\text{rect}(t)$

Note that $u(t)$ is equal to its Fourier series except at $t = -1/4$ and $t = 1/4$.

However the function $u(t)$ is not really equal to its Fourier series.

As engineers we feel this isn't a big deal.

Two functions are said to be $\mathcal{L}_2$ equivalent if their difference has zero energy, i.e.,

$$\int_{-\infty}^{\infty} |u(t) - v(t)|^2 \, dt = 0$$

$u(t)$ above is $\mathcal{L}_2$ equivalent to its Fourier series.

Two functions with the same Fourier series are also $\mathcal{L}_2$ equivalent.

Not all time-limited functions have Fourier series, even in the sense of $\mathcal{L}_2$ equivalence.

It is important for us to make general statements about whether functions have Fourier series (and similarly about Fourier integrals).

An important class of such functions are the finite energy functions, i.e., functions that are square integrable.

All physical waveforms have finite energy, but their models do not necessarily have finite energy.

Neither unit impulses nor constant functions have finite energy.

**Theorem:** **Let** $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ **be a time-limited** $\mathcal{L}_2$ **function. Then for each** $k \in \mathbb{Z}$**, the Lebesgue integral**

$$\hat{u}_k = \frac{1}{T}\int_{-T/2}^{T/2} u(t)e^{-2\pi ikt/T}\,dt$$

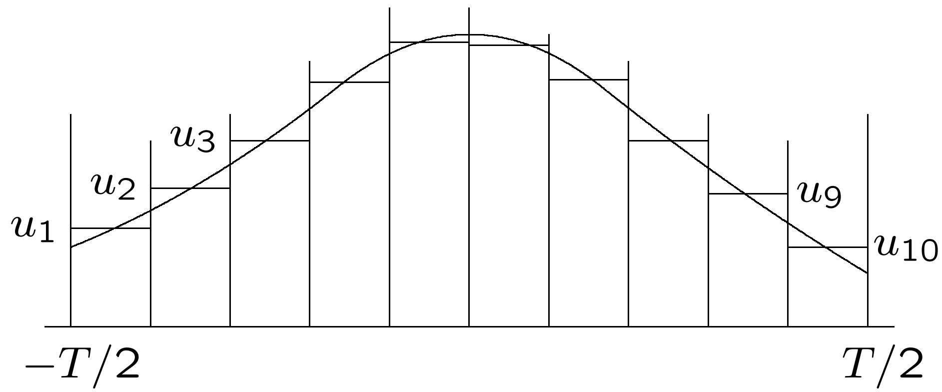
**exists as a finite complex number. Furthermore,**

$$\lim_{k_0\to\infty}\int_{-T/2}^{T/2}\left|u(t) - \sum_{k=-k_0}^{k_0}\hat{u}_k e^{2\pi ikt/T}\right|^2\,dt = 0.$$

**Also, the energy equation is satisfied. Finally, if** $\{\hat{u}_k; k \in \mathbb{Z}\}$ **is a sequence of complex numbers satisfying** $\sum_{k=-\infty}^{\infty}|\hat{u}_k|^2 < \infty$**, then an** $\mathcal{L}_2$ **function**

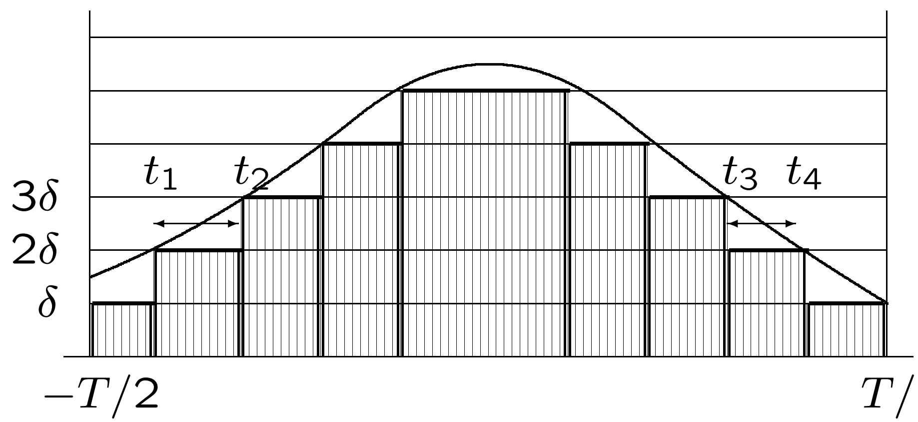
$$\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$$

**exists satisfying the above.**




$$\int_{-T/2}^{T/2} u(t)\,dt \approx \sum_{m=1}^{m_0} u_m \frac{T}{m_0} \quad \text{with } m_0 \text{ intervals}$$

**(a): Riemann**




$$\mu_2 = (t_2 - t_1) + (t_4 - t_3)$$
$$\mu_1 = (t_1 + \tfrac{T}{2}) + (\tfrac{T}{2} - t_4)$$
$$\mu_0 = 0$$

$$\int_{-T/2}^{T/2} u(t)\,dt \approx \sum_m m\delta\,\mu_m$$

**(b): Lebesgue**

Whenever the Riemann integral exists (i.e., the limit exists), the Lebesgue integral also exists and has the same value.

The familiar rules for calculating Riemann integrals also apply for Lebesgue integrals.

For some very weird functions, the Lebesgue integral exists, but the Riemann integral does not.

There are also exceptionally weird functions for which not even the Lebesgue integral exists.

The tricky thing about Lebesgue integration is the measure $\mu_m = \{t : m\delta \leq u(t) < (m{+}1)\delta\}$. This is called Lebesgue measure.

For any real $a \leq b$, including $a = -\infty$, $b = \infty$, the interval $I = (a, b)$ has measure $\mu(I) = b - a$.

Same if either or both end points included.

The measure of a finite union, $I_1, \dots, I_k$ of disjoint intervals is the sum of the measure of those intervals, $\sum_{j=1}^{k} \mu(I_j)$.

The measure of a countable union $I_1, I_2, \dots,$ of disjoint intervals is $\mu(\bigcup I_j) = \lim_{k\to\infty} \sum_{j=1}^{k} \mu(I_j)$.

**Example: What is $\mu(\mathbb{Q})$ where $\mathbb{Q}$ is the set of rationals in $[0, 1]$?**

$\mathbb{Q}$ **is countable since it can be ordered**

$$\begin{aligned} a_1 &= 1/2,\ a_2 = 1/3,\ a_3 = 2/3,\ a_4 = 1/4, \\ a_5 &= 3/4,\ a_6 = 1/5,\ a_7 = 2/5,\ \cdots . \end{aligned}$$

**Note that the point $a_j$ is the same as the closed interval $[a_j, a_j]$, which has measure 0. Thus**

$$\mu(\mathbb{Q}) = \lim_{k\to\infty} \sum_{j=1}^{k} (a_j - a_j) = 0.$$

**By the same argument, any countable set of real numbers has zero measure.**

**We need more than countable unions of intervals for a viable integration theory.**

**If $\mathcal{B}$ is measurable, we also need its complement, $\overline{\mathcal{B}}$, to be measurable with $\mu(\overline{\mathcal{B}}) = T - \mu(\mathcal{B})$.**

**Define the outer measure $\mu^{\circ}(\mathcal{A})$ of any set $\mathcal{A}$ as**

$$\mu^{\circ}(\mathcal{A}) = \inf_{\mathcal{B}\colon \mathcal{B} \text{ covers } \mathcal{A}} \mu(\mathcal{B}).$$

**where $\mathcal{B}$ covers $\mathcal{A}$ if $\mathcal{B}$ is a countable union of intervals and $\mathcal{A} \subseteq \mathcal{B}$.**

**Definition: A set $\mathcal{A}$ (over $[-T/2, T/2]$) is measurable if $\mu^{\circ}(\mathcal{A}) + \mu^{\circ}(\overline{\mathcal{A}}) = T$. If $\mathcal{A}$ is measurable, then its measure, $\mu(\mathcal{A})$, equals $\mu^{\circ}(\mathcal{A})$.**

**Each measurable set has a measurable complement.**

**If $\mathcal{A} \subset \mathcal{B}$ are both measurable, then $\mu(\mathcal{A}) \le \mu(\mathcal{B})$.**

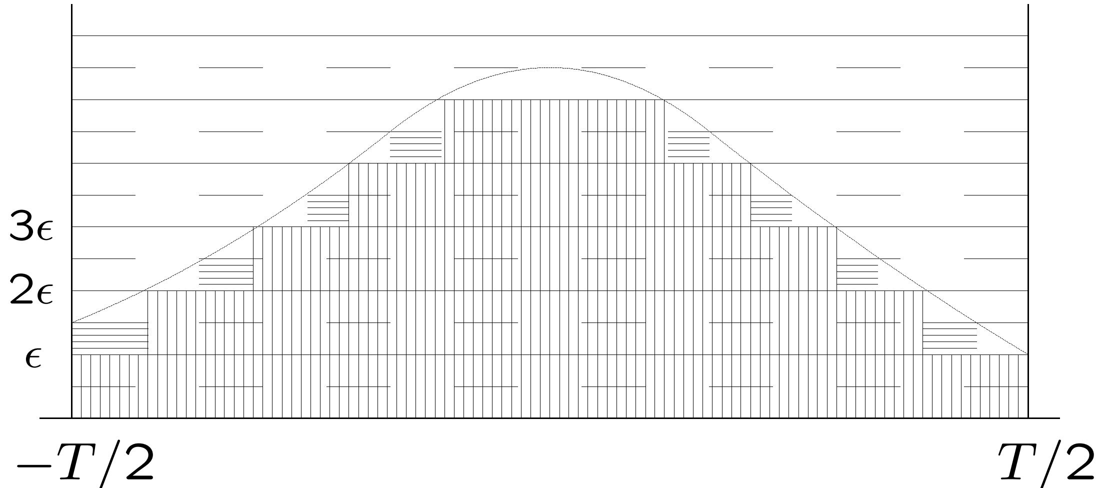
**Any measurable set can be approximated arbitrarily closely by a cover.**

**Theorem: Let $\mathcal{A}_1, \mathcal{A}_2, \ldots,$ be any sequence of measurable sets. Then $\mathcal{S} = \bigcup_{k=1}^{\infty} \mathcal{A}_k$ and $\mathcal{D} = \bigcap_{k=1}^{\infty} \mathcal{A}_k$ are measurable. If $\mathcal{A}_1, \mathcal{A}_2, \ldots$ are also disjoint, then $\mu(\mathcal{S}) = \sum_k \mu(\mathcal{A}_k)$. If $\mu^{\text{o}}(\mathcal{A}) = 0$, then $\mathcal{A}$ is measurable with measure 0.**

# MEASURABLE FUNCTIONS

**A function $\{u(t) : \mathbb{R} \to \mathbb{R}\}$ is measurable if $\{t : u(t) < b\}$ is measurable for each $b \in \mathbb{R}$.**

**The Lebesgue integral exists if the function is measurable and if the limit in the figure exists.**




**Horizontal crosshatching is what is added when $\varepsilon \to \varepsilon/2$. For $u(t) \geq 0$, the integral must exist (with perhaps an infinite value).**

**Example:** **Let** $\{h(t) : (0,1) \to \mathbb{R}$ **satisfy** $h(t) = 1$ **for** $t$ **rational,** $h(t) = 0$ **otherwise.**

**Then** $\mu(\{t : a \leq h(t) < b\})$ **is one for** $a \leq 0, b > 0$ **and is zero otherwise.**

**Thus** $\int h(t)\, dt = 0$**.**

**In other words, sets of measure zero do not effect the integral.**

**The Riemann integral does not exist for** $h(t)$**.**

**The ability to ignore sets of measure 0 is almost alone enough to justify Lebesgue integration.**

**For a positive and negative function $u(t)$ define a positive and negative part:**

$$u^+(t) = \begin{cases} u(t) & \textbf{for } t : u(t) \geq 0 \\ 0 & \textbf{for } t : u(t) < 0 \end{cases}$$

$$u^-(t) = \begin{cases} 0 & \textbf{for } t : u(t) \geq 0 \\ -u(t) & \textbf{for } t : u(t) < 0. \end{cases}$$

$$u(t) = u^+(t) - u^-(t).$$

**If $u(t)$ is measurable, then $u^+(t)$ and $u^-(t)$ are also and can be integrated as before.**

$$\int u(t) = \int u^+(t) - \int u^-(t)\,dt.$$

**except if both $\int u^+(t)\,dt$ and $\int u^-(t)\,dt$ are infinite, then the integral is undefined.**

**For $\{u(t) : [-T/2, T/2] \to \mathbb{R}\}$, the functions $|u(t)|$ and $|u(t)|^2$ are non-negative.**

**They are measurable if $u(t)$ is.**

**Their integrals exist (but might be infinite).**

$u(t)$ **is $\mathcal{L}_1$ if measurable and $\int |u(t)|\, dt < \infty$.**

$u(t)$ **is $\mathcal{L}_2$ if measurable and $\int |u(t)|^2\, dt < \infty$.**

**A complex function $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ is measurable if both $\Re[u(t)]$ and $\Im[u(t)$ are measurable.**

$\mathcal{L}_1$ **and $\mathcal{L}_2$ are defined the same way as above.**

**If** $|u(t)| \geq 1$ **for given** $t$**, then** $|u(t)| \leq |u(t)|^2$**.**

**Otherwise** $|u(t)| \leq 1$**. For all** $t$**,**

$$|u(t)| \leq |u(t)|^2 + 1.$$

**For** $\{u(t) : [-T/2, T/2 \to \mathbb{C}$**,**

$$\begin{aligned}\int_{-T/2}^{T/2} |u(t)|\,dt &\leq \int_{-T/2}^{T/2} [|u(t)|^2 + 1]\,dt \\ &= T + \int_{-T/2}^{T/2} |u(t)|^2\,dt\end{aligned}$$

**Thus** $\mathcal{L}_2$ **finite duration functions are also** $\mathcal{L}_1$**.**

**Back to Fourier series:**

**Note that** $|u(t)| = |u(t)e^{2\pi i f t}|$

**Thus, if** $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ **is** $\mathcal{L}_1$**, then**

$$\int |u(t)e^{2\pi i f t}|\, dt < \infty.$$

$$\left|\int u(t)e^{2\pi i f t}\, dt\right| \leq \int |u(t)|\, dt < \infty.$$

**If** $u(t)$ **is** $\mathcal{L}_2$ **and time-limited, it is** $\mathcal{L}_1$ **and same conclusion follows.**

**Theorem: Let $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ be an $\mathcal{L}_2$ function. Then for each $k \in \mathbb{Z}$, the Lebesgue integral**

$$\hat{u}_k = \frac{1}{T} \int_{-T/2}^{T/2} u(t)\, e^{-2\pi i k t/T}\, dt$$

**exists and satisfies $|\hat{u}_k| \le \frac{1}{T} \int |u(t)|\, dt < \infty$. Furthermore,**

$$\lim_{k_0 \to \infty} \int_{-T/2}^{T/2} \left| u(t) - \sum_{k=-k_0}^{k_0} \hat{u}_k\, e^{2\pi i k t/T} \right|^2 dt = 0,$$

**where the limit is monotonic in $k_0$.**

**We refer to this type of convergence as convergence in the mean, l.i.m. Thus the Fourier series is written as**

$$
\begin{aligned}
u(t) &= \text{l.i.m.} \sum_k \hat{u}_k \, e^{2\pi ikt/T} \, \text{rect}(\frac{t}{T}), \qquad\qquad \textbf{where} \\
\hat{u}_k &= \frac{1}{T}\int_{-T/2}^{T/2} u(t) e^{-2\pi ikt/T} \, dt, \qquad\qquad -\infty < k < \infty.
\end{aligned}
$$

**We can segment an arbitrary $\mathcal{L}_2$ function into segments of width $T$. The $m$th segment has a Fourier series:**

$$
\begin{aligned}
u_m(t) &= \text{l.i.m.} \sum_k \hat{u}_{k,m} \, e^{2\pi ikt/T} \, \text{rect}(\frac{t}{T} - m), \quad \textbf{where} \\
\hat{u}_{k,m} &= \frac{1}{T}\int_{-\infty}^{\infty} u(t) e^{-2\pi ikt/T} \text{rect}(\frac{t}{T} - m) \, dt, \;\; -\infty < k < \infty.
\end{aligned}
$$

**This breaks $u(t)$ into a double sum expansion of orthogonal functions, first over segments, then over frequencies.**

$$u(t) = \text{l.i.m.} \sum_{m,k} \hat{u}_{k,m} \, e^{2\pi ikt/T} \, \text{rect}(\frac{t}{T} - m)$$

**This is the first of a number of orthogonal expansions of arbitrary $\mathcal{L}_2$ functions.**

**It is the conceptual basis for voice compression algorithms.**

**It matches our intuition about frequency well.**

**Fourier transform:** $u(t) : \mathbb{R} \to \mathbb{C}$ **to** $\hat{u}(f) : \mathbb{R} \to \mathbb{C}$

$$\hat{u}(f) = \int_{-\infty}^{\infty} u(t) e^{-2\pi i f t} \, dt.$$

$$u(t) = \int_{-\infty}^{\infty} \hat{u}(f) e^{2\pi i f t} \, df.$$

**For "well-behaved functions," first integral exists for all $f$, second exists for all $t$ and results in original $u(t)$.**

**What does well-behaved mean? It means that the above is true.**

$$
\begin{aligned}
au(t) + bv(t) &\leftrightarrow a\hat{u}(f) + b\hat{v}(f). && \textbf{Linearity} \\
u^*(-t) &\leftrightarrow \hat{u}^*(f). && \textbf{Conjugate} \\
\hat{u}(t) &\leftrightarrow u(-f). && \textbf{Duality} \\
u(t-\tau) &\leftrightarrow e^{-2\pi i f\tau}\hat{u}(f) && \textbf{Time shift} \\
u(t)e^{2\pi i f_0 t} &\leftrightarrow \hat{u}(f - f_0) && \textbf{Frequency shift} \\
u(t/T) &\leftrightarrow T\,\hat{u}(fT). && \textbf{Scaling} \\
du(t)/dt &\leftrightarrow i2\pi f\hat{u}(f). && \textbf{Differentiation} \\
\int_{-\infty}^{\infty} u(\tau)v(t-\tau)\,d\tau &\leftrightarrow \hat{u}(f)\hat{v}(f). && \textbf{Convolution} \\
\int_{-\infty}^{\infty} u(\tau)v^*(\tau - t)\,d\tau &\leftrightarrow \hat{u}(f)\hat{v}^*(f). && \textbf{Correlation}
\end{aligned}
$$

**Two useful special cases of any Fourier transform pair are:**

$$u(0) = \int_{-\infty}^{\infty} \hat{u}(f)\,df;$$

$$\hat{u}(0) = \int_{-\infty}^{\infty} u(t)\,dt.$$

**Parseval's theorem:**

$$\int_{-\infty}^{\infty} u(t)v^*(t)\,dt = \int_{-\infty}^{\infty} \hat{u}(f)\hat{v}^*(f)\,df.$$

**Replacing** $v(t)$ **by** $u(t)$ **yields the energy equation,**

$$\int_{-\infty}^{\infty} |u(t)|^2\,dt = \int_{-\infty}^{\infty} |\hat{u}(f)|^2\,df.$$

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