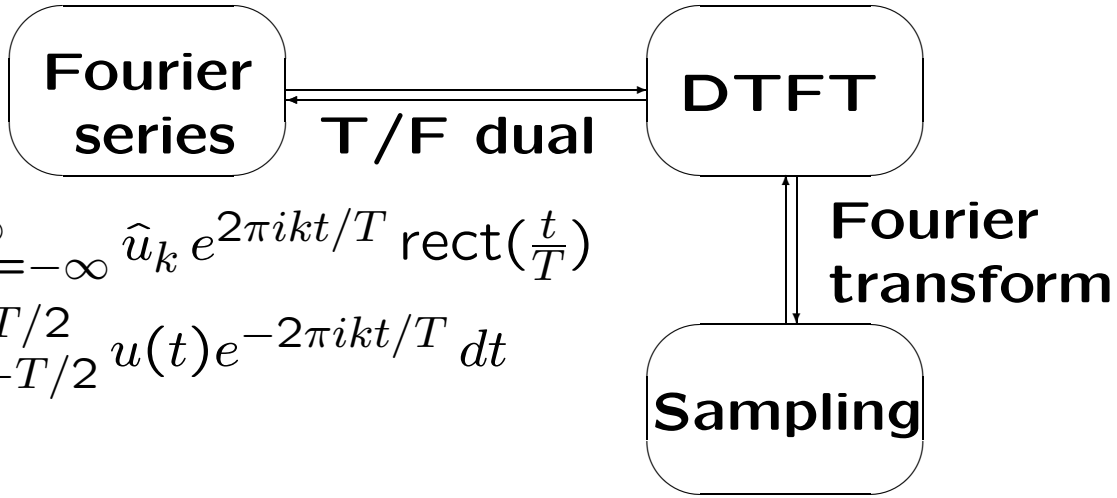
$$\widehat{u}(f) = \sum_k u_k e^{-2\pi i k \frac{f}{2W}} \mathsf{rect}\left(\frac{f}{2W}\right)$$

$$u_k = \frac{1}{2W} \int_{-W}^{W} \widehat{u}(f) e^{2\pi i k \frac{f}{2W}} \, df$$



$$u(t) = \sum_{k=-\infty}^{\infty} \widehat{u}_k \, e^{2\pi i k t/T} \, \mathsf{rect}(\tfrac{t}{T})$$

$$\widehat{u}_k = \frac{1}{T} \int_{-T/2}^{T/2} u(t) e^{-2\pi i k t/T} \, dt$$

$$u(t) = \sum_{k=-\infty}^{\infty} 2W u_k \, \mathsf{sinc}(2Wt - k)$$

$$u_k = \frac{1}{2W} u(\tfrac{k}{2W})$$

The DTFT generalizes to arbitrary frequency intervals $[\Delta - W, \Delta + W]$,

$$\widehat{v}(f) = \text{l.i.m.} \sum_k v_k e^{-2\pi i k f/(2W)} \,\text{rect}\left(\frac{f-\Delta}{2W}\right) \quad \textbf{where}$$

$$v_k = \frac{1}{2W} \int_{\Delta-W}^{\Delta+W} \widehat{v}(f) e^{2\pi i k f/(2W)} \, df.$$

Taking the inverse Fourier transform,

$$\begin{aligned} v(t) &= \sum \frac{1}{2W} v_k \,\text{sinc}(2Wt - k) \exp\{2\pi i \Delta (t - \frac{k}{2W})\} \\ &= \sum v(\frac{k}{2W}) \,\text{sinc}(2Wt - k) \exp\{2\pi i \Delta (t - \frac{k}{2W})\} \\ &= \sum v(kT) \,\text{sinc}(\frac{t}{T} - k) \exp\{2\pi i \Delta (t - kT)\} \end{aligned}$$

Just as we segmented an arbitrary $\mathcal{L}_2$ time function into intervals of duration T, we can segment an arbitrary $\mathcal{L}_2$ frequency function into intervals of duration $2W$.

$$\widehat{u}(f) = \text{l.i.m.} \sum_m \widehat{v}_m(f); \qquad \widehat{v}_m(f) = \widehat{u}(f)\,\text{rect}(\frac{f}{2W} - m).$$

$\widehat{v}_m(f)$ **is non-zero over** $[2Wm - W, 2Wm + W]$**, so**

$$\begin{aligned} v_m(t) &= \sum v_m(kT)\,\text{sinc}(\frac{t}{T} - k)\,\text{exp}\{2\pi i(2Wm)(t - kT)\} \\ &= \sum v_m(kT)\,\text{sinc}(\frac{t}{T} - k)\,\text{exp}\{2\pi imt/T\} \\ u(t) &= \sum_{k,m} v_m(kT)\,\text{sinc}(\frac{t}{T} - k)\,\text{exp}\{2\pi imt/T\} \end{aligned}$$

This is the T-spaced sinc-weighted sinusoid expansion,

$$u(t) = \text{l.i.m.} \sum_{m,k} v_m(kT)\,\text{sinc}\left(\frac{t}{T} - k\right)\, e^{2\pi imt/T}.$$

Both this and the T-spaced truncated sinusoid expansion

$$u(t) = \text{l.i.m.} \sum_{m,k} \hat{u}_{k,m}\, e^{2\pi ikt/T}\text{rect}\left(\frac{t}{T} - m\right)$$

break the function into increments of time duration T and frequency duration $1/T$.

Consider a large time interval T_0 (i.e., $[-T_0/2, T_0/2]$) and a baseband limited band W_0 (i.e., $[-W_0, W_0]$). There are T_0/T segments of duration T and W_0T positive frequency segments.

Counting negative frequencies also, there are $2T_0W_0$ time/frequency blocks and $2T_0W_0$ coefficients.

If one ignores coefficients outside of T_0, W_0, then the function is specified by $2T_0W_0$ complex numbers.

For real functions, it is T_0W_0 complex numbers.

ALIASING

Suppose we approximate a function $u(t)$ that is not quite baseband limited by the sampling expansion $s(t) \approx u(t)$.

$$s(t) = \sum_k u(kT)\,\mathrm{sinc}\left(\frac{t}{T} - k\right).$$

$$u(t) = \text{l.i.m.} \sum_{m,k} v_m(kT)\,\mathrm{sinc}\left(\frac{t}{T} - k\right)\, e^{2\pi i m t/T}$$

$$s(kT) = u(kT) = \sum_m v_m(kT) \quad \textbf{(Aliasing)}$$

$$s(t) = \sum_k \sum_m v_m(kT)\,\mathrm{sinc}\left(\frac{t}{T} - k\right).$$

$$u(t) - s(t) = \sum_k \sum_m v_m(kT) \left[e^{2\pi i m t/T} - 1 \right] \mathrm{sinc}\left(\frac{t}{T} - k \right)$$

$$= \sum_k \sum_{m \neq 0} v_m(kT) \left[e^{2\pi i m t/T} - 1 \right] \mathrm{sinc}\left(\frac{t}{T} - k \right)$$

$$\int \left| u(t) - s(t) \right|^2 dt = T \sum_k \left| \sum_{m \neq 0} v_m(kT) \right|^2 + T \sum_k \sum_{m \neq 0} \left| v_m(kT) \right|^2 .$$

$s(t)$ need not be $\mathcal{L}_2$ and first term above can be infinite.

We will see later that for a random process $U(t)$, the expected value of the first term is often the same as the second term.

ALIASING VIEWED IN FREQUENCY

Since $u(kT) = \sum_k v_m(kT)$**,**

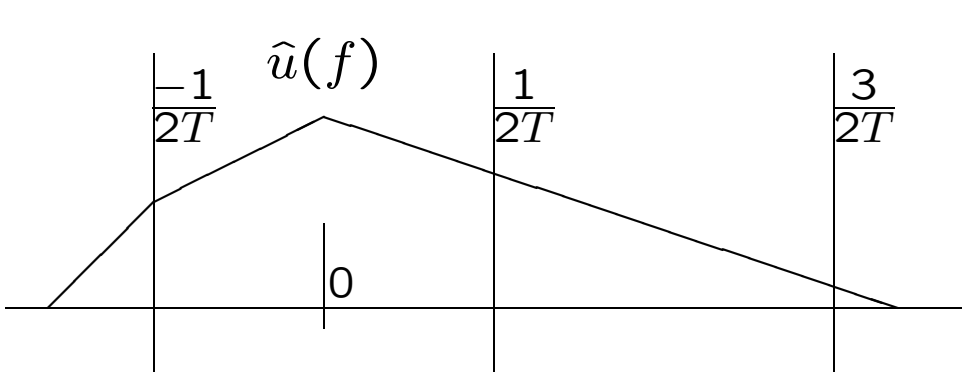
$$s(t) = \sum_m s_m(t), \qquad s_m(t) = \sum_k v_m(kT)\mathsf{sinc}\left(\frac{t}{T} - k\right).$$

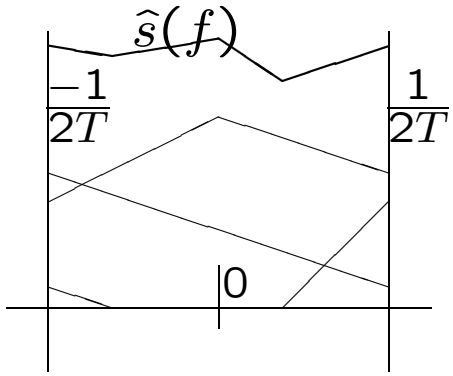
$$v_m(t) = \sum_k v_m(kT)\,\mathsf{sinc}(\frac{t}{T} - k)\,e^{2\pi imt/T}$$

$$\widehat{u}(f)\,\mathsf{rect}(fT - m) = \widehat{v}_m(f) = \widehat{s}_m(f - \frac{m}{T})$$

$$\widehat{s}(f) = \sum_m \widehat{u}(f + \frac{m}{T})\,\mathsf{rect}[fT].$$

The frequency slices are added together at baseband, losing their identity.





Theorem: Let $\widehat{u}(f)$ be $\mathcal{L}_2$, and satisfy

$$\lim_{|f|\to\infty} \widehat{u}(f)|f|^{1+\varepsilon} = 0 \qquad \textbf{for } \varepsilon > 0.$$

Then $\widehat{u}(f)$ is $\mathcal{L}_1$, and the inverse transform $u(t)$ is continuous and bounded. For $T > 0$, the sampling approx. $s(t) = \sum_k u(kT)\,\mathrm{sinc}(\frac{t}{T} + k)$ is bounded and continuous. $\widehat{s}(f)$ satisfies

$$\widehat{s}(f) = \text{l.i.m.} \sum_m \widehat{u}(f + \frac{m}{T})\,\mathrm{rect}[fT].$$

$\mathcal{L}_2$ AS A VECTOR SPACE

We have been developing the ability to go back and forth between waveforms and sequences.

You are familiar with the use of vectors to represent n-tuples. Representing a countably infinite sequence is a small conceptual extension.

Viewing waveforms as vectors is a larger conceptual extension. We have to view vectors as abstract objects rather than as n-tuples.

Orthogonal expansions are best viewed in vector space terms.

Axioms of vector space

Addition: For each $\vec{v} \in \mathcal{V}$ and $\vec{u} \in \mathcal{V}$, there is a vector $\vec{v} + \vec{u} \in \mathcal{V}$ called the sum of $\vec{v}$ and $\vec{u}$ satisfying

1. **Commutativity: $\vec{v} + \vec{u} = \vec{u} + \vec{v}$,**

2. **Associativity: $\vec{v} + (\vec{u} + \vec{w}) = (\vec{v} + \vec{u}) + \vec{w}$ for each $\vec{v}, \vec{u}, \vec{w} \in \mathcal{V}$,**

3. **There is a unique vector $0 \in \mathcal{V}$ such that $\vec{v} + 0 = \vec{v}$ for all $\vec{v} \in \mathcal{V}$,**

4. **For each $\vec{v} \in \mathcal{V}$, there is a unique vector $-\vec{v}$ such that $\vec{v} + (-\vec{v}) = 0$.**

Every vector space $\mathcal{V}$, along with the vectors, has another set of objects called scalars. For us, scalars are either the set of real or complex numbers, giving rise to real vector spaces and complex vector spaces.

Scalar multiplication: For each scalar α and each $\vec{v} \in \mathcal{V}$ there is a vector $\alpha\vec{v} \in \mathcal{V}$ called the product of α and $\vec{v}$ satisfying

1. **Scalar associativity: $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$ for all scalars α, β, and all $\vec{v} \in \mathcal{V}$,**

2. **Unit multiplication: for the unit scalar 1, $1\vec{v} = \vec{v}$ for all $\vec{v} \in \mathcal{V}$.**

Distributive laws:

1. **For all scalars α and all $\vec{v}, \vec{u} \in \mathcal{V}$,** $\alpha(\vec{v} + \vec{u}) = \alpha\vec{v} + \alpha\vec{u}$**;**

2. **For all scalars α, β and all $\vec{v} \in \mathcal{V}$,** $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$**.**

In the axiomatic approach, one establishes results from the axioms, and anthing that satisfies the axioms is a vector space and satisfies those results.

Elementary view: Vectors are n-tuples of real or complex numbers.

For real vector space, tuples and scalars are real.

For complex vector space, tuples and scalars are complex

Real 2D or 3D vectors are points in plane or space and extremely valuable for visualization.

Even complex 2D space is hard to visualize.

Simple properties:

Cancellation: if $\vec{u} + \vec{v} = \vec{w} + \vec{v}$, then $\vec{u} = \vec{w}$

Multiplication by 0: $0\vec{v} = \vec{0}$

Subtraction: $\vec{u} - \vec{v}$ means $\vec{u} + (-\vec{v})$

Solving equations: $\vec{u} - \vec{v} = \vec{w} \implies \vec{u} = \vec{v} + \vec{w}$

Note: scalars can't be infinite; neither can vectors.

For space $\mathcal{L}_2$ of finite energy complex functions, we can define $\vec{u}+\vec{v}$ as function $\vec{w}$ where $w(t) = u(t) + v(t)$ for each t.

Define $\alpha\vec{v}$ as vector $\vec{u}$ for which $u(t) = \alpha v(t)$.

Theorem: $\mathcal{L}_2$ is a complex vector space.

Proof: One question: For $\vec{v}, \vec{u} \in \mathcal{L}_2$, is $\vec{u}+\vec{v} \in \mathcal{L}_2$?

i.e., is it true that $\int_{-\infty}^{\infty} |u(t)+v(t)|^2 \; dt < \infty$

For each t $|u(t)+v(t)|^2 \leq 2|u(t)|^2 + 2|v(t)|^2$; thus yes.

Inner product spaces

Vector space definition lacks distance and angles.

Inner product adds these features. The inner product of $\vec{v}$ and $\vec{u}$ is denoted $\langle \vec{v}, \vec{u} \rangle$. Axioms:

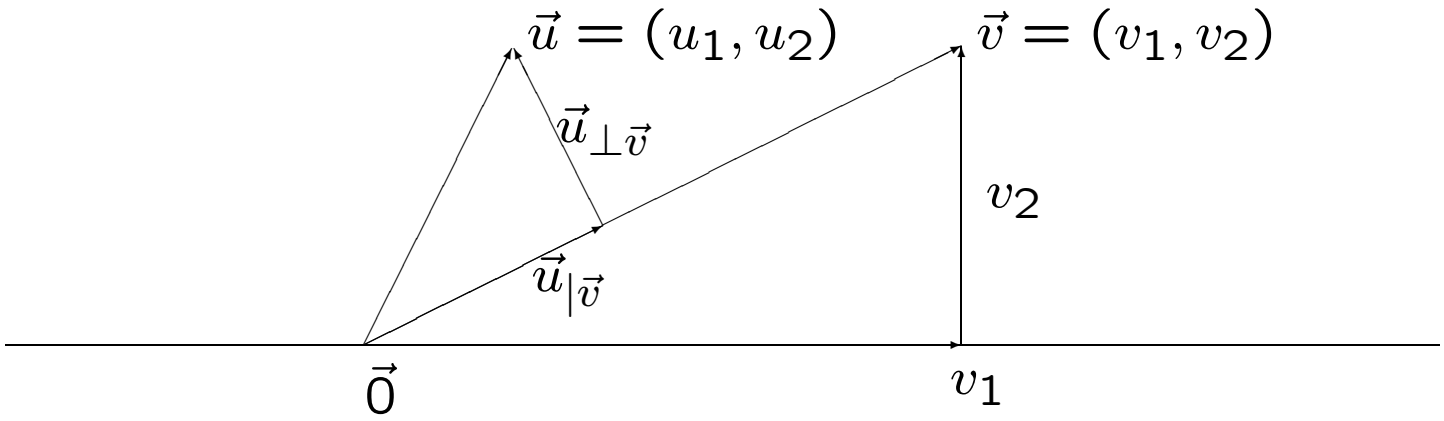
1. **Hermitian symmetry:** $\langle \vec{v}, \vec{u} \rangle = \langle \vec{u}, \vec{v} \rangle^*$
2. **Hermitian bilinearity:** $\langle \alpha\vec{v}+\beta\vec{u}, \vec{w} \rangle = \alpha\langle \vec{v}, \vec{w} \rangle + \beta\langle \vec{u}, \vec{w} \rangle$
3. **Strict positivity:** $\langle \vec{v}, \vec{v} \rangle \geq 0$**, equality iff** $\vec{v} = \vec{0}$**.**

For $\mathbb{C}^n$, we usually define $\langle \vec{v}, \vec{u} \rangle = \sum_i v_i u_i^*$.

If $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ are unit vectors in $\mathbb{C}^n$, then $\langle \vec{v}, \vec{e}_i \rangle = v_i$.

Definitions: $\|\vec{v}\|^2 = \langle \vec{v}, \vec{v} \rangle$ **is squared norm of** $\vec{v}$. $\|\vec{v}\|$ **is length of** $\vec{v}$. $\vec{v}$ **and** $\vec{u}$ **are orthogonal if** $\langle \vec{v}, \vec{u} \rangle = 0$.

More generally $\vec{u}$ **can be broken into a part** $\vec{u}_{\perp \vec{v}}$ **that is orthogonal to** $\vec{v}$ **and another part collinear with** $\vec{v}$.



Theorem: (1D Projection) Let $\vec{v}$ and $\vec{u} \neq 0$ be arbitrary vectors in a real or complex inner product space. Then there is a unique scalar α for which $\langle \vec{v} - \alpha\vec{u}, \vec{u} \rangle = 0$. That α is given by $\alpha = \langle \vec{v}, \vec{u} \rangle / \|\vec{u}\|^2$.

Proof: Calculate $\langle \vec{v} - \alpha\vec{u}, \vec{u} \rangle$ for an arbitrary scalar α and find the conditions under which it is zero:

$$\langle \vec{v} - \alpha\vec{u}, \vec{u} \rangle = \langle \vec{v}, \vec{u} \rangle - \alpha \langle \vec{u}, \vec{u} \rangle = \langle \vec{v}, \vec{u} \rangle - \alpha \|\vec{u}\|^2,$$

which is equal to zero if and only if $\alpha = \langle \vec{v}, \vec{u} \rangle / \|\vec{u}\|^2$.

Pythagorean theorem: For $\vec{v}, \vec{u}$ orthogonal,

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Thus

$$\|\vec{u}\|^2 = \|\vec{u}_{|\vec{v}}\|^2 + \|\vec{u}_{\perp\vec{v}}\|^2$$

It follows that $\|\vec{u}\|^2 \geq \|\vec{u}_{|\vec{v}}\|^2 = |\alpha|^2\|\vec{u}\|^2$. This yields the Schwartz inequality,

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \, \|\vec{v}\|$$

If we define the inner product of the vector space $\mathcal{L}_2$ as

$$\langle \vec{u}, \vec{v} \rangle = \int_{-\infty}^{\infty} u(t) v^*(t) dt,$$

then $\mathcal{L}_2$ becomes an inner product space.

Because $\langle \vec{u}, \vec{u} \rangle \neq 0$ for $\vec{u} \neq 0$, we must define quality as $\mathcal{L}_2$ equivalence.

A subspace of a vector space $\mathcal{V}$ is a subset $\mathcal{S}$ of $\mathcal{V}$ that forms a vector space in its own right.

Equivalent: For all $\vec{u}, \vec{v} \in \mathcal{S}$, $\alpha\vec{u} + \beta\vec{v} \in \mathcal{S}$

Important: $\mathbb{R}^n$ is not a subspace of $\mathbb{C}^n$; real $\mathcal{L}_2$ is not a subspace of complex $\mathcal{L}_2$.

A subspace of an inner product space (using the same inner product) is an inner product space.

The set $\vec{v}_1, \ldots, \vec{v}_n$ is linearly dependent if $\sum_{j=1}^{n} \alpha_j \vec{v}_j = 0$ for some set of scalars that are not all equal to 0.

The set is linearly independent if it is not dependent.

The dimension of a space (or subspace) is the largest number of indepentent vectors in that space.

Any set of n linearly independent vectors in an n dimensional space are said to be a basis of that space.

A set of vectors spans $\mathcal{V}$ if all vectors in $\mathcal{V}$ are linear combinations of that set.

Theorem: For n dimensional space, every basis spans space and no set of fewer than n vectors spans space.

6.450 Principles of Digital Communication I
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