Flat Rayleigh fading

Assume a single tap model with $G_{0,m} = G_m$. Assume G_m is circ. symmetric Gaussian with $\mathbf{E}[|G_m|^2] = 1$.

The magnitude is Rayleigh with

$$f_{|G_m|}(|g|) = 2|g|\exp\{-|g|^2\} \qquad ; \qquad |g| \geq 0$$
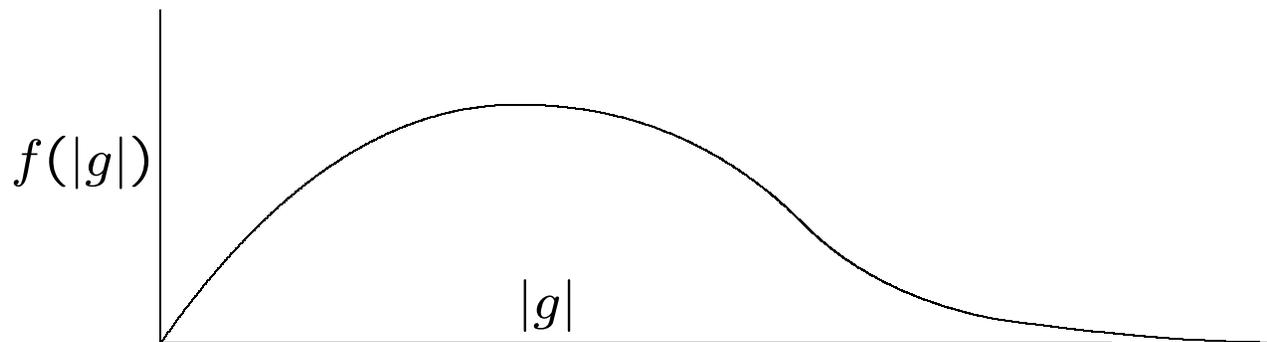
$$\Re(G_m) \sim \mathcal{N}(0, 1/2); \qquad \Im(G_m) \sim \mathcal{N}(0, 1/2)$$

$$V_m = U_m G_m + Z_m; \quad \Re(Z_m), \Im(Z_m) \sim \mathcal{N}(0, W N_0/2)$$

Antipodal binary communication does not work here. It can be viewed as phase modulation (180°); the phase of V_m is independent of U_m.

We could use binary modulation with $U_m{=}0$ or $U_m{=}a$; this is awkward and performs badly.

Consider pulse position modulation over 2 samples.

$$H = 0 \longrightarrow (U_0, U_1) = (a, 0)$$
$$H = 1 \longrightarrow (U_0, U_1) = (0, a).$$

This is equivalent to any binary scheme which
uses one or the other of 2 symmetric complex
degrees of freedom.

$$H = 0 \quad \longrightarrow \quad V_0 = aG_0 + Z_0; \quad V_1 = Z_1$$
$$H = 1 \quad \longrightarrow \quad V_0 = Z_0; \quad V_1 = aG_1 + Z_1.$$

$$H = 0 \quad \longrightarrow \quad V_0 \sim \mathcal{N}_c(0, a^2 + WN_0); \quad V_1 \sim \mathcal{N}_c(0, WN_0)$$
$$H = 1 \quad \longrightarrow \quad V_0 \sim \mathcal{N}_c(0, WN_0); \quad V_1 \sim \mathcal{N}_c(0, a^2 + WN_0).$$

$\mathcal{N}_c(0, \sigma^2)$ means iid real, imaginary, each $\mathcal{N}(0, \sigma^2/2)$.

$$H = 0 \quad \longrightarrow \quad V_0 \sim \mathcal{N}_c(0,\, a^2 + WN_0); \quad V_1 \sim \mathcal{N}_c(0,\, WN_0)$$
$$H = 1 \quad \longrightarrow \quad V_0 \sim \mathcal{N}_c(0,\, WN_0); \quad V_1 \sim \mathcal{N}_c(0,\, a^2 + WN_0).$$

$$f(v_0, v_1 | H{=}0) \;=\; \alpha\, \exp\left\{ -\frac{|v_0|^2}{a^2 + WN_0} - \frac{|v_1|^2}{WN_0} \right\}$$

$$f(v_0, v_1 | H{=}1) \;=\; \alpha\, \exp\left\{ -\frac{|v_0|^2}{WN_0} - \frac{|v_1|^2}{a^2 + WN_0} \right\}$$

$$LLR(v_0, v_1) \;=\; \ln\left\{ \frac{f(v_0, v_1 | H_0)}{f(v_0, v_1 | H_1)} \right\} \;=\; \frac{\left[|v_0|^2 - |v_1|^2 \right] a^2}{(a^2 + WN_0)(WN_0)}$$

Given $H = 0$, $|V_0|^2$ is exponential, mean $a^2 + WN_0$ and $|V_1|^2$ is exponential, mean WN_0. Error if the sample value for the first less than that of the second.

Let $X_0 = |V_0|^2$ **and** $X_1 = |V_1|^2$. **Given** H_0, X_1 **is an exponential rv with mean** WN_0 **and** X_0 **is exponential with mean** $WN_0 + a^2$.

$$\Pr(e \,|\, H = 0) = \Pr(X_0 < X_1 \,|\, H = 0).$$

The text use tedious calculation to show that

$$\Pr(e|H_0) \;=\; \frac{WN_0}{a^2 + 2WN_0} = \frac{1}{2 + a^2/(WN_0)}$$

More elegant approach: consider a Poisson process with arrivals at rate λ. **Segment time into tiny intervals of width** δ. **The probability of an arrival in an interval is** $\delta\lambda$; **arrivals are independent.**

The probability of no arrival for n intervals is

$$(1 - \delta\lambda)^n = \exp(n \ln(1 - \delta\lambda) \approx \exp(-n\delta\lambda).$$

The probability of no arrival for an interval y is $\exp(-\lambda y)$. The time to the first arrival is an exponential rv Y_0 with $f_{Y_0}(y) = \lambda \exp(-\lambda y)$.

Consider another independent Poisson process with rate γ. The time to the first arrival here is exponential with $f_{Y_1}(y) = \gamma \exp(-\gamma y)$.

The sum of these processes is Poisson with rate $\lambda + \gamma$. Each arrival is from the first process with probability $\lambda/(\lambda + \gamma)$. Thus

$$\Pr(Y_0 < Y_1) = \frac{\lambda}{\lambda + \gamma}$$

$$\Pr(Y_0 < Y_1) = \frac{\lambda}{\lambda + \gamma} = \frac{1}{1 + \gamma/\lambda}$$

For the exponential rv's X_0 and X_1 conditional on $H{=}0$ in the Rayleigh fading case, $\lambda = 1/(a^2 + WN_0)$ and $\gamma = 1/(WN_0)$. Thus

$$\Pr(X_0 {<} X_1 | H{=}0) = \frac{1}{1 + \frac{a^2 + WN_0}{WN_0}} = \frac{1}{2 + a^2/(WN_0)}$$

This is the probability of error given $H{=}0$, and by symmetry, the probability of error given $H{=}1$. The signal power is $a^2/2$ and there are $W/2$ bits per second. Thus $E_b = a^2/W$ and

$$\Pr(e) = \frac{1}{2 + E_b/N_0}$$

We next look at non-coherent detection. We use the same model except to assume that $|g_0| = |g_1| = \tilde{g}$ is known. We calculate $\Pr(e)$ conditional on $\tilde{g}$.

We find that knowing $\tilde{g}$ does not aid detection. We also find that the Rayleigh fading result occurs because of the fades rather than lack of knowledge about them.

$$H = 0 \quad \longrightarrow \quad V_0 = a\tilde{g}e^{i\phi_0} + Z_0; \quad V_1 = Z_1$$
$$H = 1 \quad \longrightarrow \quad V_0 = Z_0; \quad V_1 = a\tilde{g}e^{i\phi_1} + Z_1.$$

The phases are independent of H, so $|V_0|$ and $|V_1|$ are sufficient statistics.

The ML decision is $\tilde{H} = 0$ if $|V_0| \geq |V_1|$. This decision does not depend on $\tilde{g}$.

Since the phase of G and that of the noise are independent, we can choose rectangular coordinates with real G. The calculation is straightforward but lengthy.

$$\Pr(e) = \frac{1}{2} \exp\left(\frac{-a^2 \tilde{g}^2}{2W N_0}\right) = \frac{1}{2} \exp\left(\frac{-E_b}{2N_0}\right)$$

If the phase is known at the detector,

$$\Pr(e) = Q\left(\frac{a^2 \tilde{g}^2}{W N_0}\right) \leq \sqrt{\frac{N_0}{2\pi E_b}} \exp\left(\frac{-E_b}{2N_0}\right)$$

When the exponent is large, the db difference in E_b to get equality is small.

If $\Pr(e)$ **(incoherent result) is averaged over fading, we get flat Rayleigh fading result.**

Rayleigh fading result and incoherent result based on sending one bit at a time and using 2 complex degrees of freedom to send that bit.

For that case, knowing the channel amplitude does not help (the incoherent receiver does not use it).

Knowing the channel phase helps a little but not much.

Diversity

Consider a two tap model. More generally consider independent observations of the input.

Consider the input $H{=}0 \rightarrow a, 0, 0, 0$ and $H{=}1 \rightarrow 0, 0, a, 0$.

For $H{=}0$, $\vec{V}' = aG_{0,0}, aG_{1,1}, 0, 0$. For $H{=}1$, $\vec{V}' = 0, 0, aG_{0,2}, aG_{1,3}$.

Assume each G is $\mathcal{N}_c(0,1)$ and each Z is $\mathcal{N}_c(0,\sigma^2)$. Given $H=0$, V_1 and V_2 are $\mathcal{N}_c(0,a^2+\sigma^2)$ and V_2,V_3 are $\mathcal{N}_c(0,\sigma^2)$.

Given $H=1$, V_1 and V_2 are $\mathcal{N}_c(0,\sigma^2)$ and V_2,V_3 are $\mathcal{N}_c(0,a^2+\sigma^2)$.

Sufficient statistic is $|V_j|^2$ for $1 \le j \le 4$. Even simpler, $|V_1|^2 + |V_2|^2 - |V_3|^2 - |V_4|^2$ is a sufficient statistic.

$$\Pr(e) = \frac{4 + 3\frac{a^2}{\sigma^2}}{\left(2 + \frac{a^2}{\sigma^2}\right)^3} = \frac{4 + \frac{3E_b}{2N_0}}{\left(2 + \frac{E_b}{2N_0}\right)^3}$$

This goes down with $(E_b/N_0)^{-2}$ as $(E_b/N_0) \to \infty$.

CHANNEL MEASUREMENT

Channel measurement <u>is not</u> very useful at the receiver for single bit transmission in flat Rayleigh fading.

It <u>is</u> useful for modifying transmitter rate and power.

It <u>is</u> useful when diversity is available.

It <u>is</u> useful if a multitap model for channel is appropriate. This provides a type of diversity (each tap fades approximately independently).

Diversity results differ greatly depending on whether receiver knows channel and transmitter knows channel.

SIMPLE PROBING SIGNALS

Assume k_0 **channel taps,** $G_{0,m}, \ldots, G_{k_0-1,m}$**.**
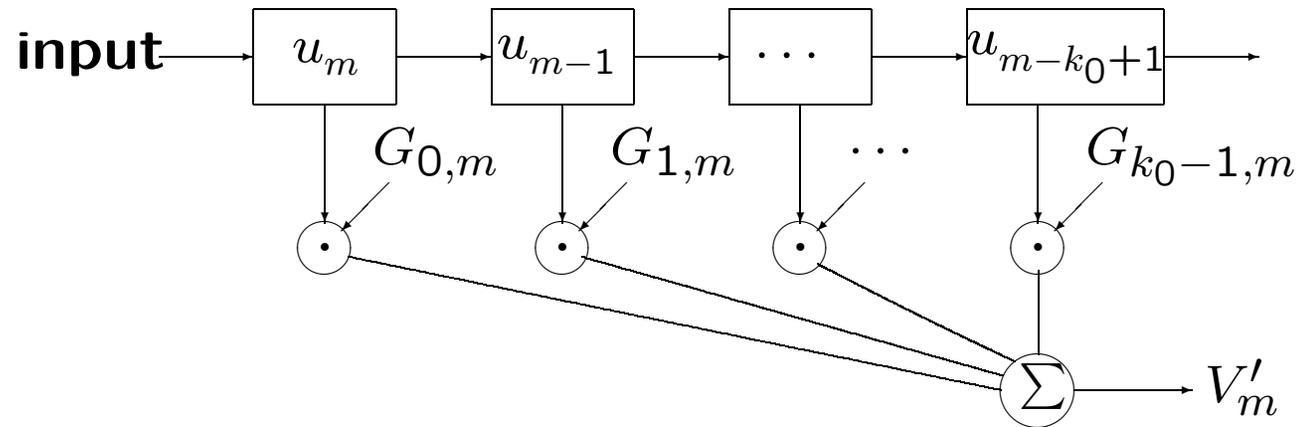
$$V'_m = u_m G_{0,m} + u_{m-1} G_{1,m} + \cdots + u_{m-k_0+1} G_{k_0-1,m}$$

Send $(a, 0, 0, \ldots, 0)$

$$\vec{V}' = (aG_{0,0}, aG_{1,1}, \ldots, aG_{k_0-1,k_0-1}, 0, 0, \ldots, 0)$$

$V_m = V'_m + Z_m$**. Estimate** $G_{m,m}$ **as** V_m/a**. Estimation error is** $\mathcal{N}_c(0, WN_0/a^2)$**.**

Pseudonoise (PN) PROBING SIGNALS

A PN sequence $\vec{u}$ is a binary sequence that appears to be iid. It is generated by a binary shift register with the mod-2 sum of given taps fed back to the input. With length k, it generates all $2^k - 1$ binary non-zero k-tuples and is periodic with length $2^k - 1$.

$$\text{PN sequence} \quad 0 \to a, 1 \to -a \xrightarrow{\quad} \boxed{\vec{G}} \xrightarrow{\vec{V}'} \overset{\displaystyle \vec{Z}\downarrow}{\oplus} \xrightarrow{\vec{V}} \boxed{\widetilde{\vec{u}}} \xrightarrow{\quad} \overset{\displaystyle \frac{1}{a^2 n}\downarrow}{\odot} \longrightarrow \vec{G} + \Psi$$
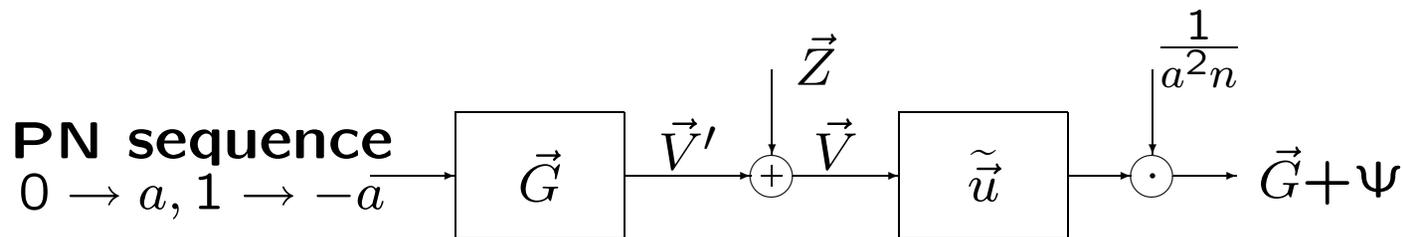
$\vec{u}$ is $\approx$ orthogonal to each shift of $\vec{u}$ so

$$\sum_{m=1}^{n} u_m u_{m+k}^* \approx \begin{cases} a^2 n & ; \quad k = 0 \\ 0 & ; \quad k \neq 0 \end{cases} \quad = \quad a^2 n \delta_k$$

If $\widetilde{\vec{u}}$ is matched filter to $\vec{u}$, then $\vec{u} * \widetilde{\vec{u}} = a^2 n \delta_j$.

Binary feedback shift register

Periodic with period $15 = 2^4 - 1$
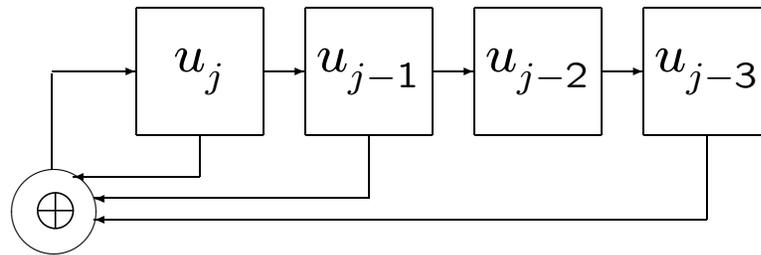
If $\vec{u} * \widetilde{\vec{u}} = a^2 n \delta_j$, then

$$\vec{V}' * \widetilde{\vec{u}} = (\vec{u} * \vec{G}) * \widetilde{\vec{u}} = (\vec{u} * \widetilde{\vec{u}} * \vec{G}) = a^2 n \vec{G}$$

The PN property has the same effect as using a single input surrounded by zeros.
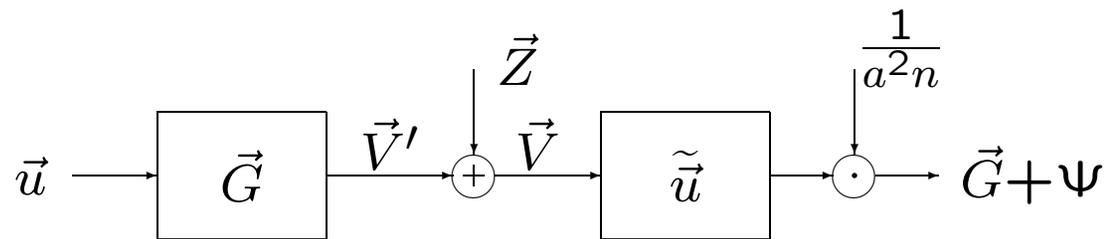
The response at time m of $\tilde{u}$ to $\vec{Z}$ is the sum of n iid rv's each of variance $a^2 N_0 W$.

The sum has variance $a^2 n N_0 W$. After scaling by $1/(a^2 n)$, $\mathsf{E}[|\Psi_k|^2] = \frac{N_0 W}{a^2 n}$.

The output is a ML estimate of $\vec{G}$; MSE decreases with n

RAKE RECEIVER

The idea here is to measure the channel and make decisions at the same time.

Assume a binary input, $H=0 \rightarrow \vec{u}^0$ and $H=1 \rightarrow \vec{u}^1$

With a known channel $\vec{g}$, the ML decision is based on pre-noise inputs $\vec{u}^0 * \vec{g}$ and $\vec{u}^1 * \vec{g}$.

$$\Re(\langle \vec{v}, \vec{u}^0 * \vec{g} \rangle) \underset{\tilde{H}=1}{\overset{\tilde{H}=0}{\underset{<}{\geq}}} \Re(\langle \vec{v}, \vec{u}^1 * \vec{g} \rangle).$$

We can detect using filters matched to $\vec{u}^0 * \vec{g}$ and $\vec{u}^1 * \vec{g}$
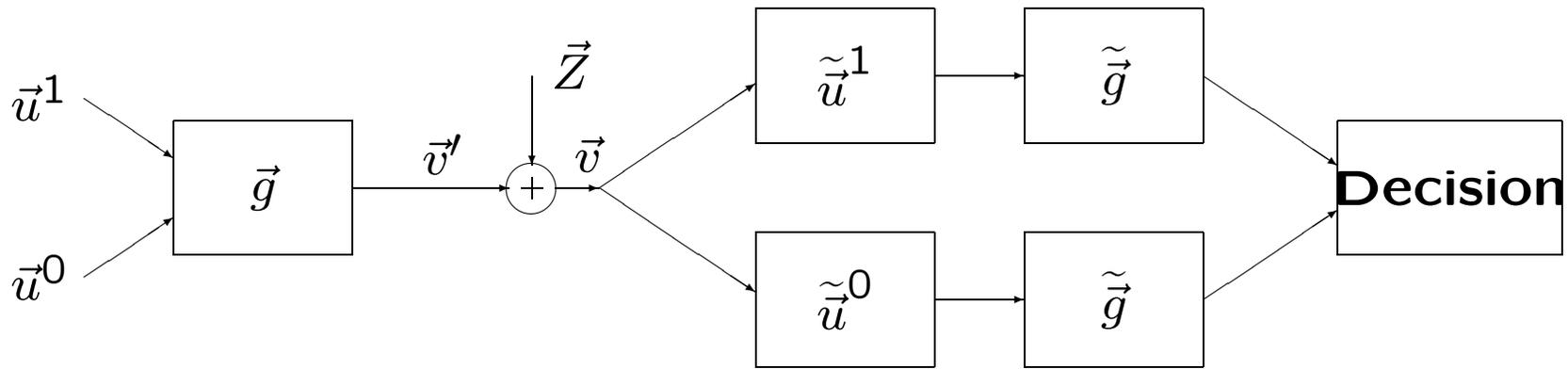
Note the similarity of this to the block diagram for measuring the channel.

If the inputs are PN sequences (which are often used for spread spectrum systems), then <u>if the correct decision</u> can be made, the output of the corresponding arm contains a measurement of $\vec{g}$.
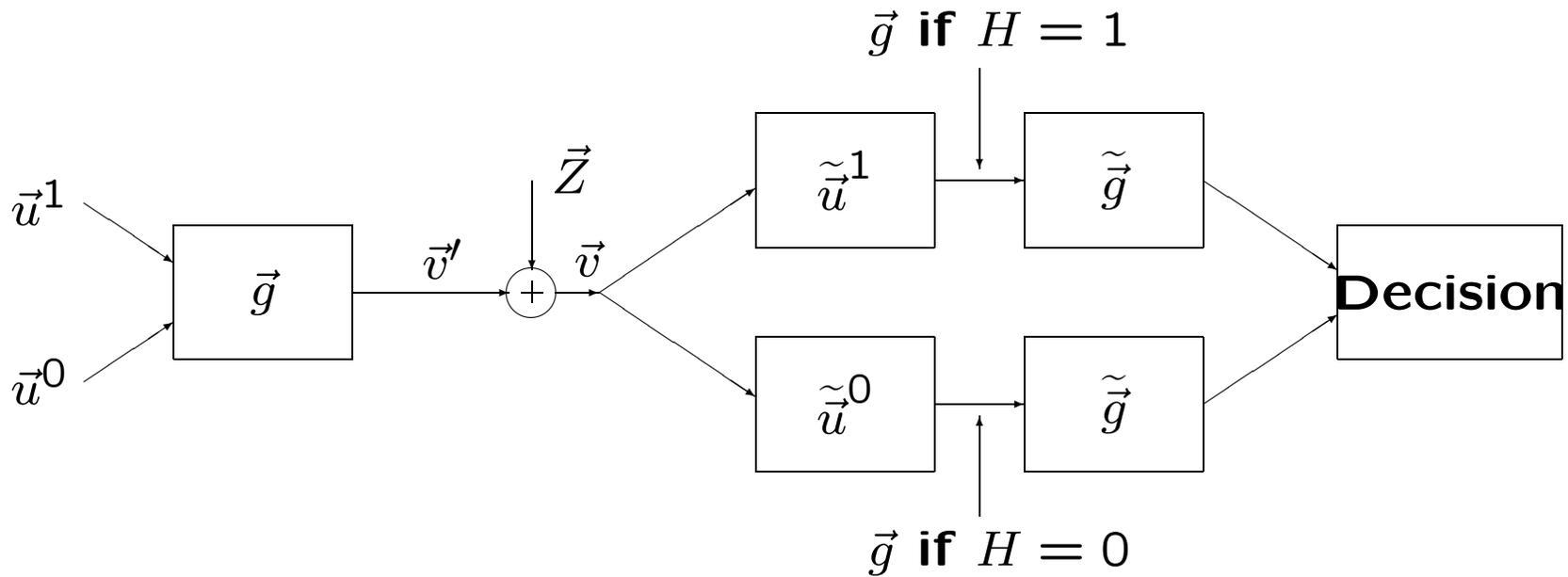
$\vec{u}^1$ and $\vec{u}^0$ are non-zero from time 1 to n. $\vec{v}'$ is non-zero from 1 to $n+k_0-1$.

$\tilde{\vec{u}}^1$ and $\tilde{\vec{u}}^0$ are non-zero from $-n$ to -1 (receiver time).

If $H = 1$ or $H = 0$, then $\vec{g}$ plus noise appears from time 0 to $k_0 - 1$ where shown. Decision is made at time 0, receiver time.

20
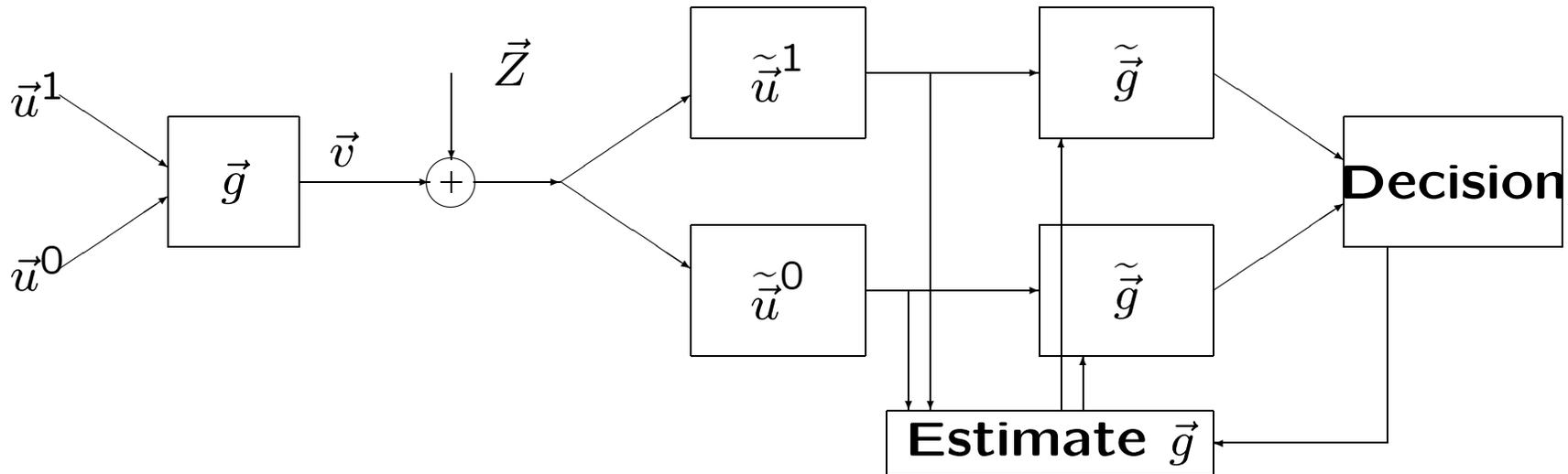
If $\hat{H} = 0$, then a noisy version of $\vec{g}$ probably exists at the output of the matched filter $\vec{u}^0$. That estimate of $\vec{g}$ is used to update the matched filters $\widetilde{\vec{g}}$.

If $\mathcal{T}_c$ is large enough, the decision updates can provide good estimates.

Suppose there is only one Rayleigh fading tap in the discrete-time model.

Suppose the estimation works perfectly and $\vec{g}$ is always known. Then the probability of error is the coherent error probability $Q(\sqrt{E_b/N_0})$ for orthogonal signals and $E_b = a^2 n|g|^2/W$.

This is smaller than incoherent $\Pr(e) = \frac{1}{2}\exp\{-E_b/(2N_0)\}$.

Averaging over G, incoherent result is $\frac{1}{2+\bar{E}_b/N_0}$ and coherent result is at most half of this.

Measurement doesn't help here.

Diversity

Consider a two tap model. More generally consider independent observations of the input.

Input
$\{a, 0\}$

U_m

U_{m-1}

$G_{0,m}$

$G_{1,m}$

Z_m

Σ

V'_m

$+$

V_m

Consider the input $H{=}0 \rightarrow a, 0, 0, 0$ and $H{=}1 \rightarrow 0, 0, a, 0$.

For $H{=}0$, $\vec{V}' = aG_{0,0}, aG_{1,1}, 0, 0$. For $H{=}1$, $\vec{V}' = 0, 0, aG_{0,2}, aG_{1,3}$.

Assume each G is $\mathcal{N}_c(0,1)$ and each Z is $\mathcal{N}_c(0,\sigma^2)$. Given $H=0$, V_1 and V_2 are $\mathcal{N}_c(0, a^2 + \sigma^2)$ and V_2, V_3 are $\mathcal{N}_c(0, \sigma^2)$.

Given $H=1$, V_1 and V_2 are $\mathcal{N}_c(0, \sigma^2)$ and V_2, V_3 are $\mathcal{N}_c(0, a^2 + \sigma^2)$.

Sufficient statistic is $|V_j|^2$ for $1 \leq j \leq 4$. Even simpler, $|V_1|^2 + |V_2|^2 - |V_3|^2 - |V_4|^2$ is a sufficient statistic.

$$\Pr(e) = \frac{4 + 3\frac{a^2}{\sigma^2}}{\left(2 + \frac{a^2}{\sigma^2}\right)^3} = \frac{4 + \frac{3E_b}{2N_0}}{\left(2 + \frac{E_b}{2N_0}\right)^3}$$

This goes down with $(E_b/N_0)^{-2}$ as $(E_b/N_0) \to \infty$.

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