QUIZ 1 SOLUTIONS

Problem 1: (30 points) *Consider a discrete memoryless source with alphabet* $\{1, 2, \ldots, M\}$*. Suppose that the symbol probabilities are ordered and satisfy* $p_1 > p_2 > \cdots > p_M$ *and also satisfy* $p_1 < p_{M-1} + p_M$*. Let* $l_1, l_2, \ldots, l_M$ *be the lengths of a prefix-free code of minimum expected length for such a source.*

a) *Show that* $l_1 \le l_2 \le \cdots \le l_M$*.*

This is almost the same as Lemma 1 in lecture 3. In particular, assume to the contrary that $p_i > p_j$ and $l_i > l_j$. By interchanging the codeword for i with that for j, the difference between the old and new $\overline{L}$ is

$$(p_i l_i + p_j l_j) - (p_i l_j + p_j l_i) = (p_i - p_j)(l_i - l_j) > 0.$$

Thus the expected length is reduced, showing that the original code is non-optimum.

b) *Show that if the Huffman algorithm is used to generate the above code, then* $l_M \le l_1 + 1$*. Hint: the easy way is to look only at the first step of the algorithm and not to use induction.*

In the reduced code after the first step of the Huffman algorithm, the codeword for the combined symbol of probability $p_M + p_{M-1}$ must have a length l' less than or equal to l_1 (this follows from part (a) plus the fact that $p_1 < p_M + p_{M-1}$). Since $l' = l_M - 1$, the result follows.

c) *Show that* $l_M \le l_1 + 1$ *whether or not the Huffman algorithm is used to generate a minimum expected length prefix-free code.*

A minimum-expected-length code must be full, and thus the codeword for letter M must have a sibling, say letter j. Since $p_j \ge p_{M-1}$, we have $p_j + p_M > p_1$. Let l' be the length of the intermediate node that is parent to j and M. Now $l' \le l_1$ since otherwise the codeword for 1 could be interchanged with this intermediate node for a reduction in $\overline{L}$. Again $l_M - 1 \le l_1$.

d) *Suppose* $M = 2^k$ *for integer* k*. Determine* $l_1, \ldots, l_M$*.*

First assume $l_1 = k$. Then all codewords have length k or $k + 1$, but the Kraft inequality can be satisfied with equality (*i.e.*, the code can be full) only if all codewords have length k. If $l_1 > k$, then $l_j > k$ for all j and the Kraft inequality can not be satisfied with equality. Finally, if $l_1 < k$ with $l_j \le k$, then the Kraft inequality can not be met. Thus all codewords have length k.

e) *Suppose* $2^k < M < 2^{k+1}$ *for integer* k*. Determine* $l_1, \ldots, l_M$*.*

This is essentially the same as problem 2.3(b) in the homework. The codewords all have length k or $k+1$. Let m be the number of length k codewords. To satisfy Kraft with equality, $m2^{-k} + (M - m)2^{-k-1} = 1$. Solving, $m = 2^{k+1} - M$.

Problem 2: (35 points) *Consider a source X with M symbols, $\{1, 2, \ldots, M\}$ ordered by probability with $p_1 \geq p_2 \geq \cdots \geq p_M > 0$. The Huffman algorithm operates by joining the two least likely symbols together as siblings and then constructs an optimal prefix-free code for a reduced source X' in which the symbols of probability p_M and p_{M-1} have been replaced by a single symbol of probability $p_M + p_{M-1}$. The expected code-length $\overline{L}$ of the code for the original source X is then equal to $\overline{L}' + p_M + p_{M-1}$ where $\overline{L}'$ is the expected code-length of X'.*

a) *Express the entropy $\mathsf{H}(X)$ for the original source in terms of the entropy $\mathsf{H}(X')$ of the reduced source as*

$$\mathsf{H}(X) = \mathsf{H}(X') + (p_M + p_{M-1})H(\gamma) \tag{1}$$

where $H(\gamma)$ is the binary entropy function, $H(\gamma) = -\gamma \log \gamma - (1-\gamma) \log(1-\gamma)$. Find the required value of γ to satisfy (2).

$$\begin{aligned}
\mathsf{H}(X) &= \sum_{i=1}^{M} -p_i \log p_i \\
\mathsf{H}(X') &= -(p_{M-1}+p_M) \log(p_{M-1}+p_M) - \sum_{i=1}^{M-2} p_i \log p_i \\
\mathsf{H}(X) - \mathsf{H}(X') &= (p_{M-1}+p_M) \log(p_{M-1}+p_M) - p_M \log p_M - p_{M-1} \log p_{M-1} \\
&= (p_{M-1}+p_M) \left[-\frac{p_M}{p_{M-1}+p_M} \log \frac{p_M}{p_{M-1}+p_M} - \frac{p_{M-1}}{p_{M-1}+p_M} \log \frac{p_{M-1}}{p_{M-1}+p_M} \right] \\
&= (p_{M-1}+p_M) H\left(\frac{p_M}{p_{M-1}+p_M} \right)
\end{aligned}$$

so $\gamma = \frac{p_M}{p_{M-1}+p_M}$.

b) *In the code tree generated by the Huffman algorithm, let v_1 denote the intermediate node that is the parent of the leaf nodes for symbols M and $M-1$. Let $q_1 = p_M + p_{M-1}$ be the probability of reaching v_1 in the code tree. Similarly, let $v_2, v_3, \ldots,$ denote the subsequent intermediate nodes generated by the Huffman algorithm. How many intermediate nodes are there, including the root node of the entire tree?*

Each step of the Huffman algorithm reduces the number of symbols by 1 until only 1 node (the root) is left. Thus there are $M - 1$ intermediate nodes, counting the root.

c) *Let $q_1, q_2, \ldots,$ be the probabilities of reaching the intermediate nodes $v_1, v_2, \ldots,$ (note that the probability of reaching the root node is 1). Show that $\overline{L} = \sum_i q_i$. Hint: Note that $\overline{L} = \overline{L}' + q_1$.*

After the second step of the algorithm, $\overline{L}'$ is related to the minimum expected length, say $\overline{L}^{(2)}$ of the further reduced code by $\overline{L}' = \overline{L}^{(2)} + q_2$. Thus $\overline{L} = \overline{L}^{2} + q_1 + q_2$. Proceeding to step $M-1$, (or more formally using induction) we have $\overline{L} = \overline{L}^{M-1} + q_1 + q_2 + \ldots, q_{M-1}$. Since the expected length for the root node is 0, $\overline{L} = \sum_i q_i$.

d) *Express $\mathsf{H}(X)$ as a sum over the intermediate nodes. The ith term in the sum should involve q_i and the binary entropy $H(\gamma_i)$ for some γ_i to be determined. You may find it helpful to define α_i as the probability of moving upward from intermediate node v_i, conditional on reaching v_i. (Hint: look at part a).*

Note that in part (a), $\mathsf{H}(X) = \mathsf{H}(X') + H(\gamma)q_1$ where $\gamma = p_M/(p_M + p_{M-1})$ is the probability of moving up (or down) in the tree at node v_1 conditional on reaching node v_1. Since $H(\gamma) = H(1-\gamma)$, we see that

$$\mathsf{H}(X) = \mathsf{H}(X') + H(\alpha_1)q_1.$$

Applying the same argument to the reduced ensemble X' in terms of the next reduction $X^{(2)}$, $\mathsf{H}(X') = \mathsf{H}(X^{(2)}) + q_2 H(\alpha_2)$. Proceeding to the root (*i.e.*, using induction),

$$\mathsf{H}(X) = \sum_{i=1}^{M-1} q_i \, H(\alpha_i)$$

e) *Find the conditions (in terms of the probabilities and binary entropies above) under which $\overline{L} = \mathsf{H}(X)$.*

$$\overline{L} - \mathsf{H}(X) = \sum_i q_i[1 - H(\alpha_i)]$$

The binary entropy $H(\gamma)$ is less than or equal to 1, reaching 1 only when $\gamma = 1/2$. Thus $\overline{L} - \mathsf{H}(X) = 0$ if and only if $\alpha_i = 1/2$ for all i. This implicitly assumes that $p_i > 0$ for all symbols.

f) *Are the formulas for $\overline{L}$ and $\mathsf{H}(X)$ above specific to Huffman codes alone, or do they apply (with the modified intermediate node probabilities and entropies) to arbitrary full prefix-free codes?*

The same arguments apply to any full code tree with any procedure for successively reducing sibling leaf nodes into the parent.

Problem 3: (35 points) *Consider a discrete source U with a finite alphabet of N real numbers, $r_1 < r_2 < \cdots < r_N$ with the pmf $p_1 > 0, \ldots, p_N > 0$. The set $\{r_1, \ldots, r_N\}$ is to be quantized into a smaller set of $M < N$ representation points $a_1 < a_2 < \cdots < a_M$.*

a) *Let $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_M$ be a given set of quantization intervals with $\mathcal{R}_1 = (-\infty, b_1], \mathcal{R}_2 = (b_1, b_2], \ldots, \mathcal{R}_M = (b_{M-1}, \infty)$. Assume that at least one source value r_i is in $\mathcal{R}_j$ for each j, $1 \le j \le M$ and give a necessary condition on the representation points $\{a_j\}$ to achieve minimum MSE.*

Each representation point a_j must be chosen as the conditional mean of the set of symbols in $\mathcal{R}_j$. Specifically,

$$a_j = \frac{\sum_{i \in \mathcal{R}_j} p_i \, r_i}{\sum_{i \in \mathcal{R}_j} p_i}.$$

The reason for this is the same as that when the source has a probability density. For the given set of regions, and the assumption that each region contains a symbol from the source alphabet, this is both necessary and sufficient. Since the regions might not be optimally chosen, however, it is only a necessary condition on the overall minimum MSE.

b) *For a given set of representation points* $a_1, \ldots, a_M$*, assume that no symbol* r_i *lies exactly halfway between two neighboring* a_j, i.e., *that* $r_i \neq \frac{a_j + a_{j+1}}{2}$ *for all* i, j*. Determine which interval* $\mathcal{R}_j$ *(and thus which representation point* a_j*) each symbol* r_i *must be mapped into to minimize MSE for the given* $\{a_j\}$*. Note that it is not necessary to place the boundary* b_j *between* $\mathcal{R}_j$ *and* $\mathcal{R}_{j+1}$ *at* $b_j = [a_j + a_{j+1}]/2$ *since there is no probability in the immediate vicinity of* $[a_j + a_{j+1}]/2$*.*

The symbol r_i has a squared error $|r_i - a_j|^2$ if mapped into $\mathcal{R}_j$ and thus into a_j. Thus r_i must be mapped into the closest a_j and thus the region $\mathcal{R}_j$ must contain all source symbols that are closer to a_j than to any other representation point. The quantization intervals are not uniquely determined by this rule since $\mathcal{R}_j$ can end and $\mathcal{R}_{j+1}$ can begin at any point between the largest source symbol closest to a_j and the smallest source symbol closest to a_{j+1}.

c) *For the given representation points,* $a_1, \ldots, a_M$*, now assume that* $r_i = \frac{a_j + a_{j+1}}{2}$ *for some source symbol* r_i *and some* j*. Show that the MSE is the same whether* r_i *is mapped into* a_j *or into* a_{j+1}*.*

Snce r_i is midway between a_j and a_{j+1}, the squared error is $|r_i - a_j|^2 = |r_i - a_{j+1}|^2$ no matter whether r_i is mapped into a_j or a_{j+1}.

d) *For the assumption in part c), show that the set* $\{a_j\}$ *cannot possibly achieve minimum MSE. Hint: Look at the optimal choice of* a_j *and* a_{j+1} *for each of the two cases of part c).*

We assume that the minimum MSE is achieved with representation points a_j and a_{j+1} such that source value $r_i = (a_j + a_{j+1})/2$ and we demonstrate a contradiction. Consider two cases, first where r_i is mapped to a_{j+1} and second where r_i is mapped to a_j.

Case 1 ($r_i \to a_{j+1}$): Let $\mathcal{R}_j$ include the points closest to a_j not including r_i and let $\mathcal{R}_{j+1}$ include the points closest to a_{j+1} including r_i.

Subcase 1.1: Assume a_j *is not* the conditional mean of $\mathcal{R}_j$. Then the MSE can be reduced by changing a_j, demonstrating the contradiction.

Subcase 1.2: Assume that a_j *is* the conditional mean of $\mathcal{R}_j$. As shown in (c), we can map r_i into a_j rather than a_{j+1} without changing the MSE. Let $\mathcal{R}'_j$ include the points closest to a_j, now including r_i. Because of the additional point in $\mathcal{R}'_j$,

$$\mathsf{E}[U|\mathcal{R}'_j] \neq a_j = \mathsf{E}[U|\mathcal{R}_j]. \tag{2}$$

(This is demonstrated below.) Thus, for the new set of intervals with $r_i \in \mathcal{R}'_j$, the MSE can be reduced by changing a_j to the new conditional mean. This establishes the contradiction for subcase 1.2 and thus for case 1.

Case 2: $(r_i \to a_j)$ The argument is the same as that in case 1, reversing the roles of j and $j+1$.

We now demonstrate (2) analytically. Let $P = \sum_{i \in \mathcal{R}_j} p_i$ and let $m = \sum_{i \in \mathcal{R}_j} p_i r_i$. Then $a_j = m/P$. Let a'_j be the conditional mean when r_i is added to $\mathcal{R}_j$. Note that r_i is greater than each source point in $\mathcal{R}_j$ and thus $r_i > a_j = m/P$. We thus have

$$a'_j = \frac{m + p_i r_i}{P + p_i} > \frac{m + p_i m/P}{P + p_i} = m \, \frac{1 + p_i/P}{P + p_i} = m/P = a_j$$