6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

6.641 — Electromagnetic Fields, Forces, and Motion Spring 2005

Final Review Packet

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Contents

Practice Problems

Problem 1

A perfectly-conducting channel of height D and width $2W$ is divided into two free-space regions by a very thin perfectly-conducting slide as shown below. The $\hat{z}$-directed magnetic fields in the left and right regions are $H_l(t)\hat{z}$ and $H_r(t)\hat{z}$, respectively. The slide has a mass M per unit length in the $\hat{z}$ direction, has a variable position $\xi(t)$, and makes a frictionless but perfectly-conducting contact with the channel.

(A) Assume that at $t = 0, H_l = H_{l_0}, H_r = H_{r_0}$, and $\xi = \xi_0$. Determine $H_l(t)$ and $H_r(t)$ in terms of H_{l_0}, H_{r_0}, ξ_0, and $\xi(t)$.

(B) The very thin slide supports a surface current $K\hat{y}$ which separates $H_l\hat{z}$ and $H_r\hat{z}$. This current interacts with the neighboring magnetic fields to produce a force $F\hat{x}$ on the slide per unit length in the $\hat{z}$ direction. Evaluate F.

(C) Use the results of (B) to find a function $V(\xi)$ so that

$$\frac{d}{dt}\left(\frac{M}{2}\left(\frac{d\xi(t)}{dt}\right)^2 + V(\xi)\right) = 0.$$

(D) At $t = 0$, $\frac{d\xi}{dt} = 0$. Determine the velocity of the slide when it first reaches $\xi(t) = 0$.

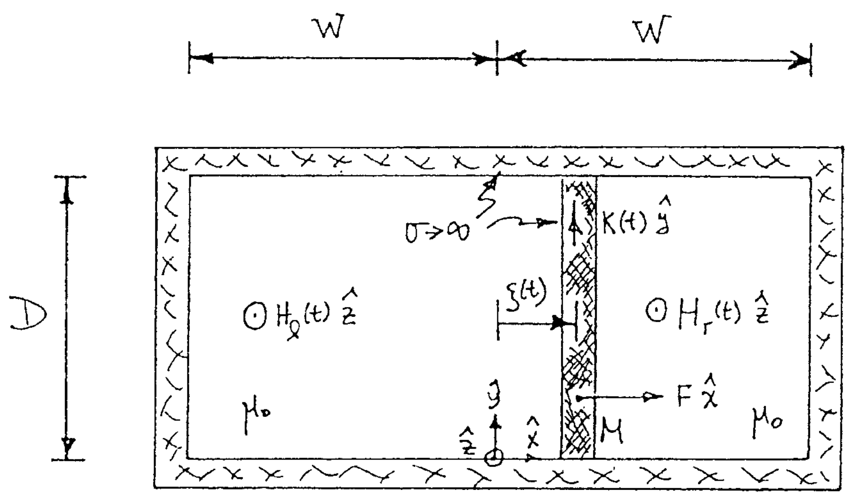

Figure 1: A perfectly-conducting channel with a perfectly conducting slide.

Problem 2

A conducting plate of thickness Δ, permeability μ_0 and conductivity σ moves with velocity U in the $\hat{z}$ direction between two perfectly-permeable plates as shown below. At $z = 0$, $\bar{B}$ is constrained to be $\text{Real}\{B_0 e^{j\omega t}\}\,\hat{x}$ by exciter coils. The perfectly-permeable plates confine $\bar{B}$ to the region $0 \le z \le L$ so that $\bar{B} = 0\hat{x}$ at $z = L$. Within the region $0 \le z \le L$, $\bar{B}$ is approximated by $\bar{B} = B_x(z,t)\hat{x}$. Ignore fringing fields.

(A) Derive a differential equation for B_x in the conducting slab for $0 \le z \le L$.

(B) Determine B_x for $0 \le z \le L$.

(C) Determine the external force f needed to pull the plate in the $\hat{z}$ direction at velocity U.

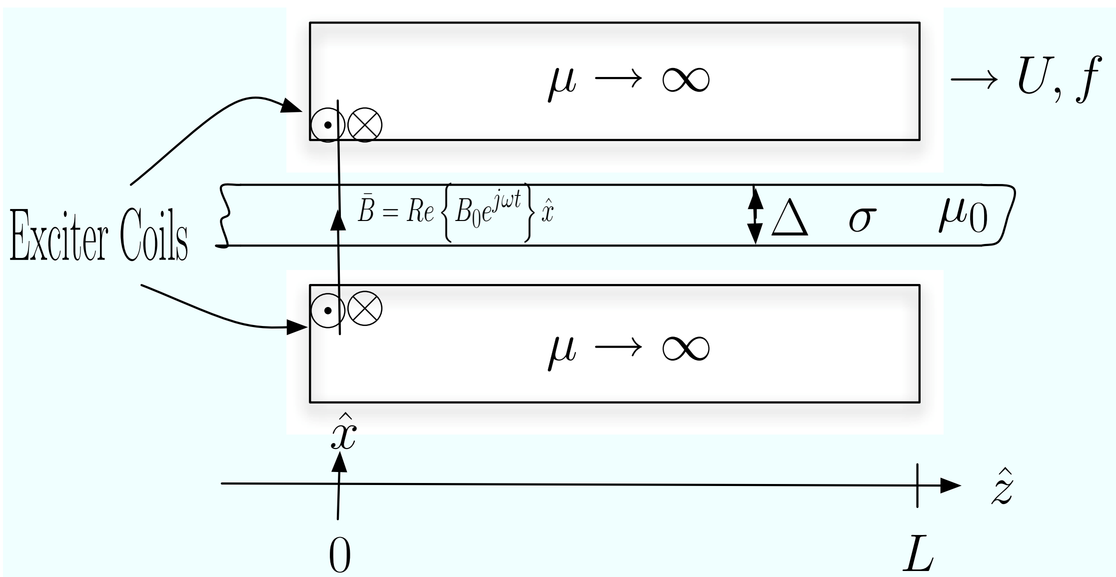

Figure 2: A conducting plate moving between infinitely permeable plates. (Image by MIT OpenCourseWare.)

Problem 3

A simplified model of a Van de Graaff generator is shown below. A belt with permittivity ϵ, conductivity σ, thickness δ, and width W travels to the right with velocity U. At $x = 0$, a charge source maintains the charge density ρ_0 on the belt. At $x = l$, a charge collector collects all the charge off the belt. An external resistance R is connected from the charge collector to the charge source. Determine the current i through the resistor when the generator is in steady state.

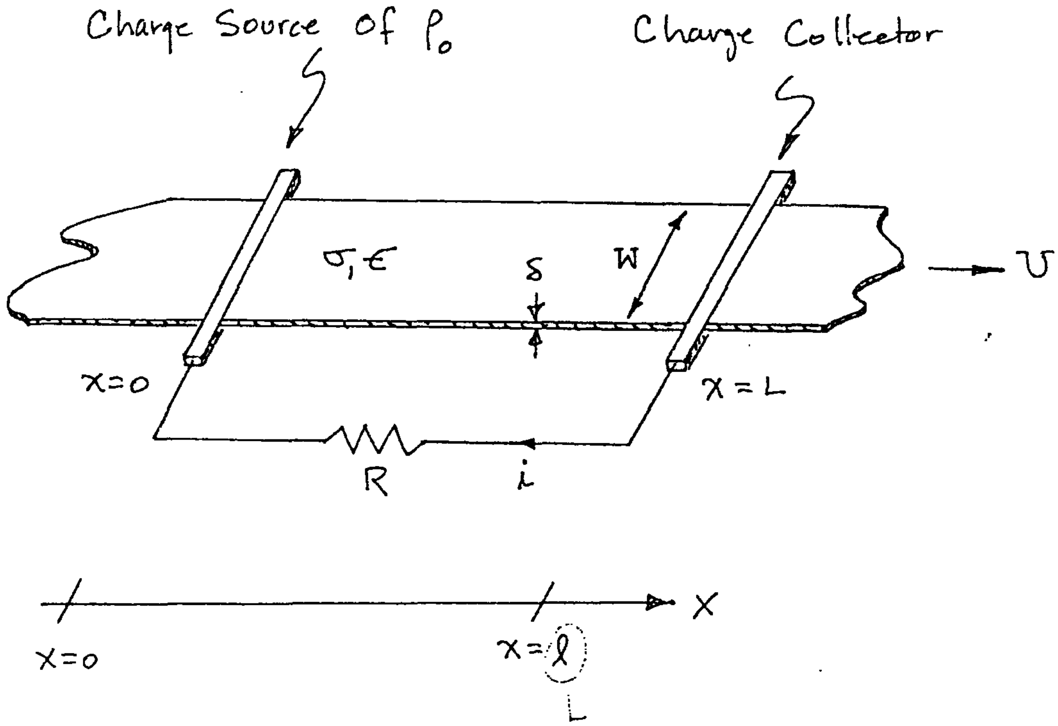

Figure 3: A simplified Van de Graaff generator.

Problem 4

A pair of grounded perfectly-conducting plates of infinite extent in the $\hat{y}$ and $\hat{z}$ directions are located at $x = \Delta$ and $x = -\Delta$ as shown below. A fluid having permittivity ϵ and conductivity σ flows with uniform velocity U in the $\hat{z}$ direction between the plates. At $t = 0$, the fluid has a charge distribution given by

$$\rho(x, y, z) = \begin{cases} \rho_0 \sin\left(\frac{\pi x}{\Delta}\right) e^{-ky^2} & |z| \leq \delta \\ 0 & |z| > \delta \end{cases}$$

Determine $\rho(x, y, z, t)$ between the plates for $t > 0$, $|x| \leq \Delta$, and all y and z.

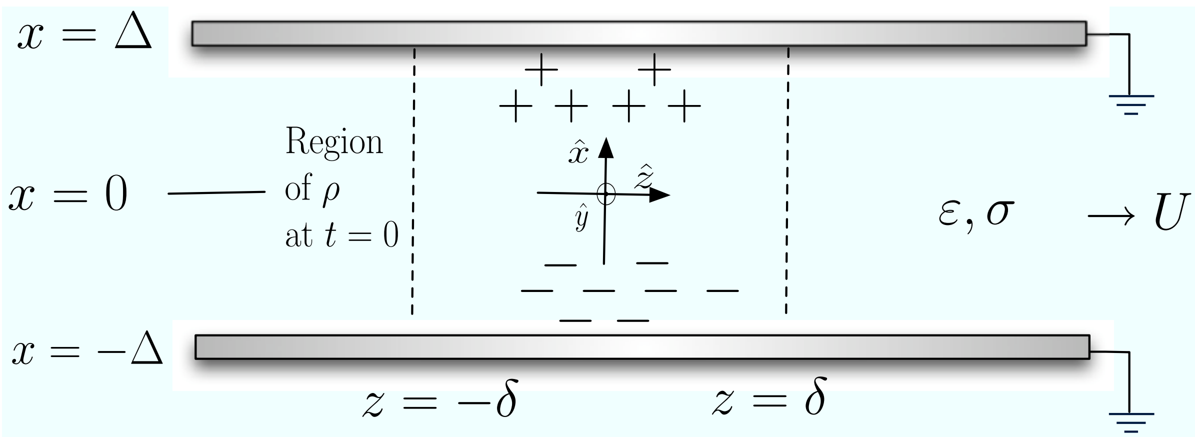

Figure 4: A pair of grounded perfectly-conducting plates enclose a moving conductor (Image by MIT OpenCourseWare.)

Problem 5

A thin sheet having effective surface conductivity σ_0 moves in the $\hat{z}$ direction with velocity U as shown below. The sheet is symmetrically located at a distance Δ between two potential sources. The potential sources are symmetrically excited as traveling waves with frequency ω and wave number k. Assume $k\Delta \ll 1$ and make appropriate assumptions.

(A) Find the electric field components E_x and E_z just above and below the thin sheet.

(B) Find the free surface charge in the thin sheet.

(C) Find the spatially and temporally averaged $\hat{z}$-directed force per $y - z$ area which acts on the sheet.

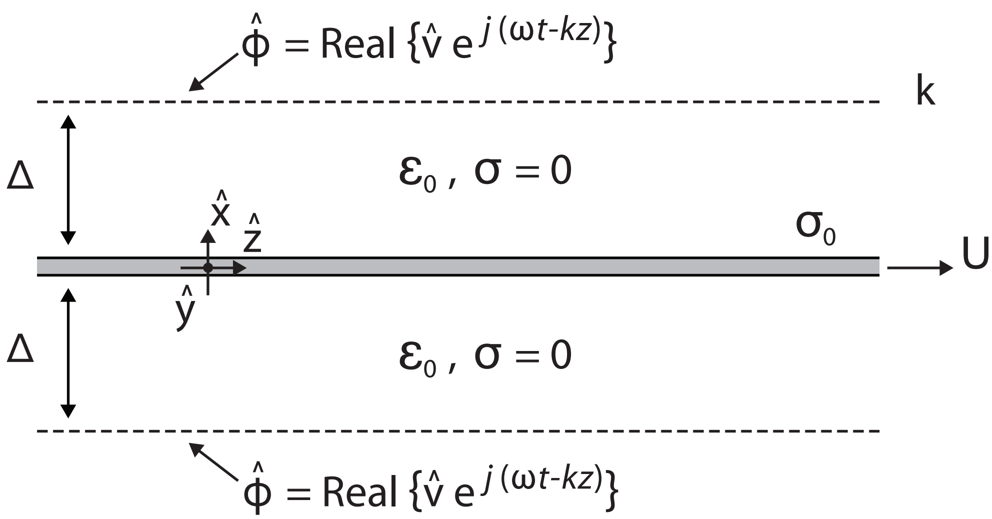

Figure 5: A thin sheet (Image by MIT OpenCourseWare.)

Final Exam 1998 Solutions

Problem 1

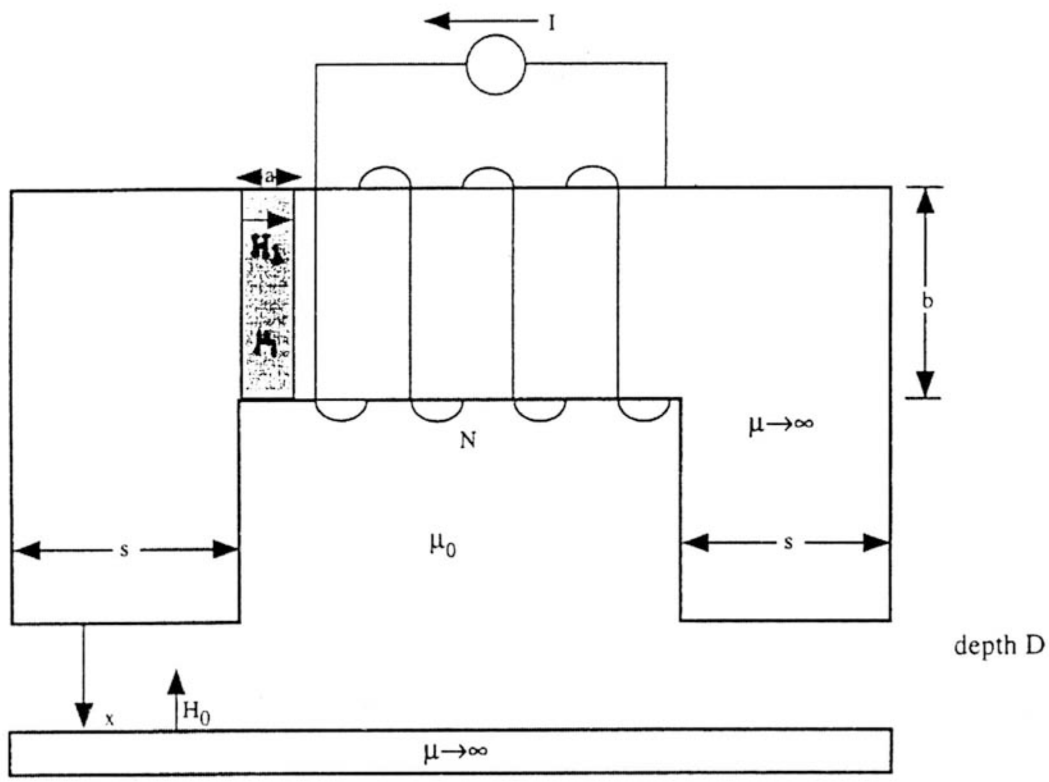

Figure 6: A magnetic circuit

In the magnetic circuit shown above, a current I flows in the N turn coil which is mounted on a material of infinite magnetic permeability ($\mu \to \infty$) except for a thin gap of width a and height b which has finite magnetic permeability μ_1. The lower plate has infinite magnetic permeability ($\mu \to \infty$) and is at a distance x below the upper assembly. The magnetic materials are surrounded by free space with magnetic permeability μ_0. The entire system has depth D.

(A) Neglecting fringing field effects, find the magnetic field H_0 in the air gap and H_1 in the thin section of the upper magnetic part.

<u>**Solution:**</u>

$$
\begin{aligned}
H_0 2x + H_1 a &= NI \\
\mu_0 H_0 s \cancel{D} &= \mu_1 H_1 b \cancel{D} \\
H_1 &= \frac{\mu_0 H_0 s}{\mu_1 b}
\end{aligned}
$$

$$H_0\left[2x+\frac{\mu_0 sa}{\mu_1 b}\right]=NI\Rightarrow H_0=\frac{NI\mu_1 b}{2\mu_1 bx+\mu_0 sa}$$
$$H_1=\frac{NI\mu_0 s}{2\mu_1 bx+\mu_0 sa}$$

(B) Find the self-inductance of the N turn coil.

Solution:

$$\begin{aligned}L&=\frac{N\Phi}{I},\Phi=\mu_0 H_0 sD=\mu_1 H_1 bD\\ &=\frac{N^2\mu_0\mu_1 bsD}{2\mu_1 bx+\mu_0 sa}\end{aligned}$$

(C) Find the total magnetic energy stored in the system.

Solution:

$$W_M=\frac{1}{2}LI^2=\frac{1}{2}\frac{(NI)^2\mu_0\mu_1 bsD}{2\mu_1 bx+\mu_0 sa}$$

Alternate Method:

$$\begin{aligned}W_M&=\frac{1}{\cancel{2}}\mu_0 H_0^2 xsD(\cancel{2})+\frac{1}{2}\mu_1 H_1^2 abD\\
&=H_0^2\left[\mu_0 xsD+\frac{1}{2}\mu_1 abD\left(\frac{\mu_0 s}{\mu_1 b}\right)^2\right]\\
&=\frac{H_0^2 D}{(\mu_1 b)^2}\left[\mu_0 xs(\mu_1 b)^2+\frac{1}{2}\mu_1 ab(\mu_0 s)^2\right]\\
&=\frac{H_0^2 D}{(\mu_1 b)^2}\mu_0\mu_1 bs\left[x\mu_1 b+\frac{1}{2}\mu_0 as\right]\\
&=\frac{(NI)^2(\mu_1^{\cancel{2}}b^{\cancel{2}})D}{\cancel{(\mu_1 b)^2}(2\mu_1 bx+\mu_0 sa)^2}\mu_0\cancel{\mu_1 b}s\left[x\mu_1 b+\frac{1}{2}\mu_0 as\right]\\
&=\frac{\frac{1}{2}(NI)^2\mu_1\mu_0 bsD}{(2\mu_1 bx_+\mu_0 sa)}\end{aligned}$$

(D) Find the magnetic force on the moveable lower plate as a function of x, material properties, N, I, and geometric dimensions.

Solution:

$$\begin{aligned}f_x&=\frac{1}{2}I^2\frac{dL}{dx}\\
&=-\frac{1}{\cancel{2}}\frac{I^2N^2\mu_0\mu_1 bsD}{(2\mu_1 bx+\mu_0 sa)^2}\cancel{2}\mu_1 b\\
&=-\frac{(NI)^2\mu_0\mu_1^2 b^2 sD}{(2\mu_1 bx+\mu_0 sa)^2}\end{aligned}$$

Problem 2

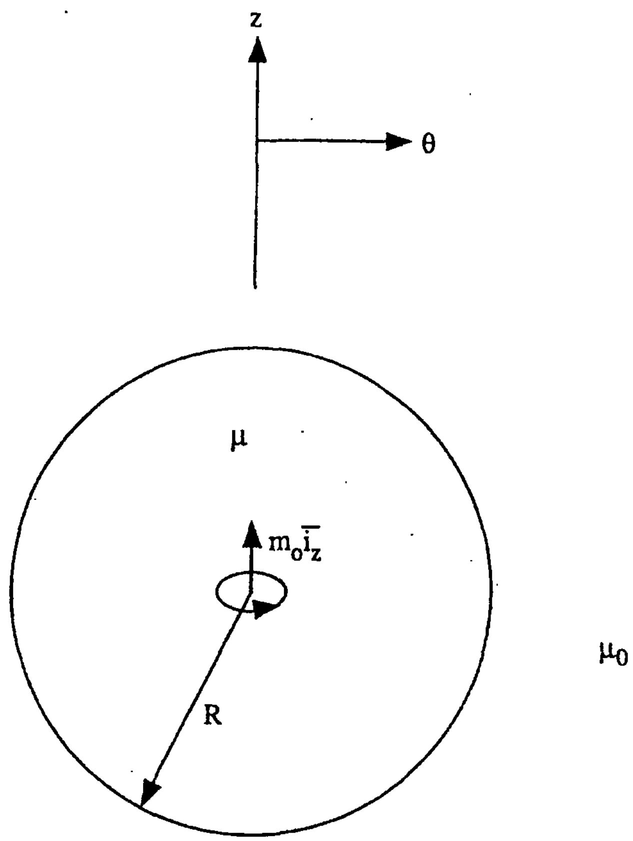

Figure 7: A sphere with a point magnetic dipole at its center

A point magnetic dipole with moment $\bar{m} = m_0\bar{i}_z$ is located at the center of a sphere of radius R. The sphere has finite magnetic permeability μ and the sphere is surrounded by free space with magnetic permeability μ_0. There is no free surface current on the $r = R$ interface.

(A) What boundary conditions must be satisfied by the magnetic scalar potential and/or magnetic field at $r = 0, r = R$, and $r = \infty$?

<u>**Solution:**</u>

$$\chi_m(r=0) = \frac{m_0}{4\pi}\frac{\cos\theta}{r^2}, \chi_m(r\to\infty) = 0$$

$$H_\theta(r=R_-) = H_\theta(r=R_+), B_r(r=R_-) = B_r(r=R_+) \Rightarrow \mu H_r(r=R_-) = \mu_0 H_r(r=R_+)$$

(B) Find the magnetic field $\bar{H}$ inside and outside the sphere.

<u>**Solution:**</u>

$$\chi_m = \begin{cases} \frac{m_0}{4\pi}\frac{\cos\theta}{r^2} + Ar\cos\theta & 0 < r < R \\ \frac{C}{r^2}\cos\theta & r > R \end{cases}$$

$$\bar{H} = -\nabla\chi_m = -\left[\frac{\partial\chi_m}{\partial r}\bar{i_r} + \frac{1}{r}\frac{\partial\chi_m}{\partial\theta}\bar{i_\theta}\right]$$

$$= \begin{cases} \frac{m_0}{4\pi r^3}\left(2\cos\theta\bar{i_r} + \sin\theta\bar{i_\theta}\right) - A\left[\cos\theta\bar{i_r} - \sin\theta\bar{i_\theta}\right] & 0 < r < R \\ \frac{C}{r^3}\left(2\cos\theta\bar{i_r} + \sin\theta\bar{i_\theta}\right) & r > R \end{cases}$$

$$H_\theta(r = R_-) = H_\theta(r = R_+) \Rightarrow \frac{C}{R^3} = \frac{m_0}{4\pi R^3} + A$$

$$\mu H_r(r = R_-) = \mu_0 H_r(r = R_+) \Rightarrow \mu\left[\frac{2m_0}{4\pi R^3} - A\right] = \frac{2\mu_0 C}{R^3}$$

$$\frac{m_0}{4\pi R^3} + A = \frac{C}{R^3}$$

$$\frac{2m_0}{4\pi R^3} - A = \frac{2\mu_0}{\mu R^3}C$$

$$\frac{C}{R^3}\left(1 + \frac{2\mu_0}{\mu}\right) = \frac{3m_0}{4\pi R^3} \Rightarrow \frac{C}{R^3} = \frac{3m_0}{4\pi R^3\left(1 + \frac{2\mu_0}{\mu}\right)}$$

$$A = \frac{C}{R^3} - \frac{m_0}{4\pi R^3} = \frac{m_0}{4\pi R^3}\left(\frac{3}{1 + \frac{2\mu_0}{\mu}} - 1\right) = \frac{m_0}{4\pi R^3}\left(\frac{2\left(1 - \frac{\mu_0}{\mu}\right)}{\left(1 + 2\frac{\mu_0}{\mu}\right)}\right)$$

$$\bar{H} = \begin{cases} \frac{m_0}{4\pi r^3}\left[2\cos\theta\bar{i_r} + \sin\theta\bar{i_\theta}\right] - \frac{m_0}{2\pi R^3}\frac{\left(1 - \frac{\mu_0}{\mu}\right)}{1 + \frac{2\mu_0}{\mu}}\left(\cos\theta\bar{i_r} - \sin\theta\bar{i_\theta}\right) & 0 < r < R \\ \frac{3m_0}{4\pi r^3}\frac{\left(2\cos\theta\bar{i_r} + \sin\theta\bar{i_\theta}\right)}{\left(1 + \frac{2\mu_0}{\mu}\right)} & \end{cases}$$

(C) What is the effective magnetic dipole moment of the sphere seen by an observer for $r > R$?

<u>**Solution:**</u>

$$\frac{m_{\text{eff}}}{4\pi} = C \Rightarrow m_{\text{eff}} = \cancel{4\pi}\frac{3m_0}{\cancel{4\pi}\left(1 + \frac{2\mu_0}{\mu}\right)} \Rightarrow m_{\text{eff}} = \frac{3m_0}{1 + \frac{2\mu_0}{\mu}}$$

Problem 3

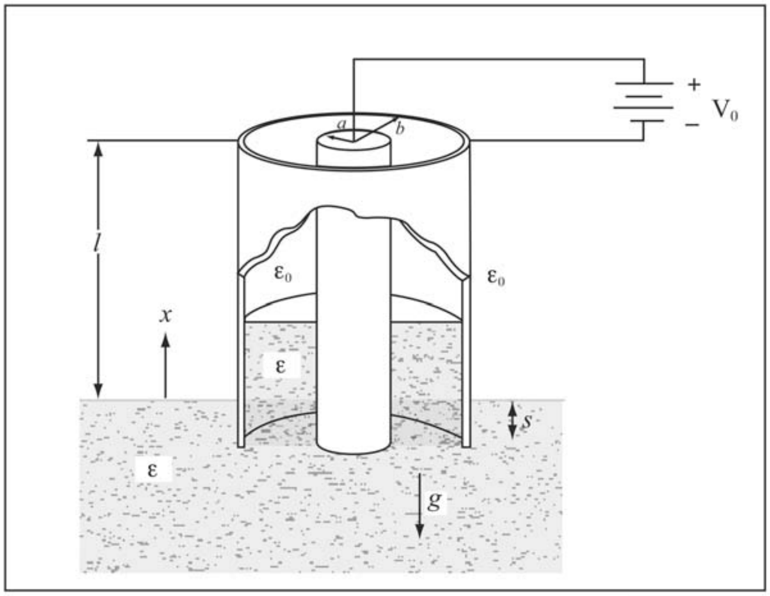

Figure 8: A coaxial cylindrical capacitor (Image by MIT OpenCourseWare.)

A coaxial cylindrical capacitor is dipped into a linearly polarizable fluid with dielectric permittivity ϵ and mass density ρ_m. Gravity is directed downwards.

When voltage V_0 is applied, the dielectric fluid is pulled into the coaxial capacitor to a height x above the fluid level outside the cylinders. If $V_0 = 0$, the fluid level within the cylinders is a distance s from the lower end of the cylinder. There is no free volume charge in the system .

(A) Neglecting fringing field effects, what is the electric field, magnitude and direction, between the cylinders $(a < r < b)$ as a function of r in both the upper free space region and in the lower dielectric fluid?

<u>**Solution:**</u>

$$\nabla \cdot \bar{E} = \frac{1}{r}\frac{d}{dr}(rE_r) = 0 \Rightarrow E_r = \frac{A}{r}$$

$$\int_a^b E_r dr = A \ln\frac{b}{a} = V_0 \Rightarrow A = \frac{V_0}{\ln\frac{b}{a}}$$

$$E_r = \frac{V_0}{r\ln\frac{b}{a}} \qquad \text{(In both regions between cylinders)}$$

Note that tangential E is continuous at the dielectric interface

(B) What is the capacitance as a function of x?

<u>**Solution:**</u>

In free space region: $D_r = \frac{\epsilon_0 V_0}{r\ln\frac{b}{a}}$.

In dielectric fluid: $D_r = \frac{\epsilon V_0}{r \ln \frac{b}{a}}$.
Free surface charge on r=a surface:

$$\begin{aligned}
q &= \epsilon_0 E_r\Big|_a 2\pi a(l-x-s) + \epsilon E_r\Big|_a 2\pi a(x+s) \\
&= E_r\Big|_a 2\pi a(\epsilon_0(l-x-s) + \epsilon(x+s)) \\
&= \frac{V_0}{\cancel{a} \ln \frac{b}{a}} 2\pi \cancel{a} \left[\epsilon_0(l-x-s) + \epsilon(x+s)\right] \\
&= \frac{2\pi V_0}{\ln \frac{b}{a}} \left[\epsilon_0(l-x-s) + \epsilon(x+s)\right]
\end{aligned}$$

$$C = \frac{q}{V_0} = \frac{2\pi \left[\epsilon_0(l-x-s) + \epsilon(x+s)\right]}{\ln \frac{b}{a}}$$

(C) What is the total electric energy stored in the system?

<u>Solution:</u>

$$W_E = \frac{1}{2} C V_0^2 = \frac{\pi \left[\epsilon_0(l-x-s) + \epsilon(x+s)\right]}{\ln \frac{b}{a}} V_0^2$$

Alternate Method:

$$\begin{aligned}
W_E &= \int_V \frac{1}{2} \epsilon E^2 dV = \int_{r=a}^{b} \int_{\Phi=0}^{2\pi} \int_{z=0}^{l} \frac{1}{2} \epsilon E^2 r dr d\Phi dz \\
W_E &= \int_{r=a}^{b} \frac{1}{2} E_r^2 r dr \left[\epsilon_0(l-x-s) + \epsilon(x+s)\right] 2\pi \\
&= \frac{\cancel{2}\pi}{\cancel{2}} \frac{\left[\epsilon_0(l-x-s) + \epsilon(x+s)\right] V_0^2}{\left[\ln(\frac{b}{a})\right]^2} \int_{r=a}^{b} \frac{dr}{r} \\
&= \frac{\pi \left[\epsilon_0(l-x-s) + \epsilon(x+s)\right] V_0^2}{\ln \frac{b}{a}}
\end{aligned}$$

(D) How high will the dielectric fluid rise when a voltage V_0 is applied?

<u>Solution:</u>

$$f_x = \frac{1}{2} V_0^2 \frac{dC}{dx} = \frac{1}{\cancel{2}} \frac{V_0^2 \cancel{2}\pi \left[\epsilon - \epsilon_0\right]}{\ln \frac{b}{a}} = \rho_m g \pi (b^2 - a^2) x$$

$$x = \frac{V_0^2 (\epsilon - \epsilon_0)}{\ln \frac{b}{a} \rho_m g (b^2 - a^2)}$$

Problem 4

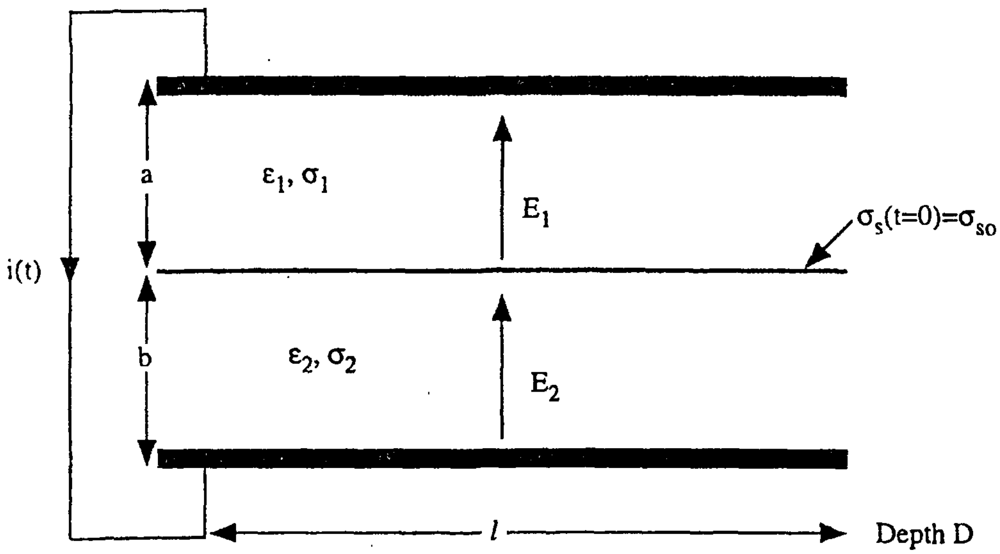

Figure 9: Two lossy dielectrics

Two lossy dielectrics with respective dielectric permittivities ϵ_1 and ϵ_2 and respective ohmic conductivities σ_1 and σ_2 are superposed within a short-circuited capacitor. At $t = 0$ there is a free surface charge density of $\sigma_{s0} \frac{C}{m^2}$ on the interface between the dielectrics. Neglect fringing field effects. The free volume charge density at time $t = 0$ is zero in both dielectrics.

(A) Find the electric fields $E_1(t = 0)$ and $E_2(t = 0)$ in both lossy dielectrics at time $t = 0$.

<u>**Solution:**</u>

$$\rho_f(t) = 0 \text{ in both dielectrics } \Rightarrow \nabla \cdot \bar{E}_1 = \nabla \cdot \bar{E}_2 = 0 \Rightarrow E_1 = E_1(t), E_2 = E_2(t)$$

$$\int_{x=0}^{a+b} E dx = \int_{x=0}^{b} E_2 dx + \int_{x=b}^{a+b} E_1 dx = E_2 b + E_1 a = 0 \Rightarrow E_2 = -\frac{E_1 a}{b}$$

at $t = 0$:

$$\epsilon_1 E_1 - \epsilon_2 E_2 = \sigma_{s0}$$

$$E_1 \left[\epsilon_1 + \frac{\epsilon_2 a}{b}\right] = \sigma_{s0}$$

$$E_1 = \frac{\sigma_{s0} b}{\epsilon_1 b + \sigma_2 a}, E_2 = -\frac{E_1 a}{b} = -\frac{\sigma_{s0} a}{\epsilon_1 b + \epsilon_2 a}$$

(B) Find the electric fields $E_1(t)$ and $E_2(t)$ in both lossy dielectrics as a function of time.

<u>**Solution:**</u>

$$\sigma_1 E_1 - \sigma_2 E_2 + \frac{d}{dt}\left[\epsilon_1 E_1 - \epsilon_2 E_2\right] = 0$$

$$E_1\left[\sigma_1 + \frac{\sigma_2 a}{b}\right] + \left(\epsilon_1 + \frac{\epsilon_2 a}{b}\right)\frac{dE_1}{dt} = 0$$

$$E_1 = E_1(t=0)e^{-\frac{t}{\tau}}; \tau = \frac{\epsilon_1 b + \epsilon_2 a}{\sigma_1 b + \sigma_2 a}$$

$$E_1 = \frac{\sigma_{s0} b}{\epsilon_1 b + \epsilon_2 a} e^{-\frac{t}{\tau}}$$

$$E_2 = -\frac{E_1 a}{b} = -\frac{\sigma_{s0} a}{\epsilon_1 b + \epsilon_2 a} e^{-\frac{t}{\tau}}$$

(C) Find the free surface charge density $\sigma_s(t)$ on the interface as a function of time.

<u>**Solution:**</u>

$$\sigma_s(t) = \epsilon_1 E_1 - \epsilon_2 E_2 = E_1\left(\epsilon_1 + \frac{\epsilon_2 a}{b}\right) = \sigma_{s0} e^{-\frac{t}{\tau}}$$

(D) Find the short circuit current $i(t)$ that flows in the wire short-circuiting the two electrodes as a function of time.

<u>**Solution:**</u>

$$\begin{aligned}
\frac{i(t)}{ld} &= \sigma_2 E_2 + \epsilon_2 \frac{dE_2}{dt} = \sigma_1 E_1 + \epsilon_1 \frac{dE_1}{dt} \\
&= \left(\sigma_1 - \frac{\epsilon_1}{\tau}\right) E_1 \\
&= \left(\sigma_1 - \frac{\epsilon_1}{(\epsilon_1 b + \sigma_2 a)}\left(\sigma_1 b + \sigma_2 a\right)\right) E_1 \\
&= \left(\frac{\cancel{\sigma_1 \epsilon_1 b} + \sigma_1 \epsilon_2 a - \cancel{\epsilon_1 \sigma_1 b} - \epsilon_1 \sigma_2 a}{\epsilon_1 b + \epsilon_2 a}\right) E_1 \\
&= \frac{a(\sigma_1 \epsilon_2 - \epsilon_1 \sigma_2)}{(\epsilon_1 b + \epsilon_2 a)^2} \sigma_{s0} b e^{-\frac{t}{\tau}} \\
i(t) &= \frac{ab\sigma_{s0}(\sigma_1 \epsilon_2 - \epsilon_1 \sigma_2) ld}{(\epsilon_1 b + \epsilon_2 a)^2} e^{-\frac{t}{\tau}}
\end{aligned}$$

Problem 5

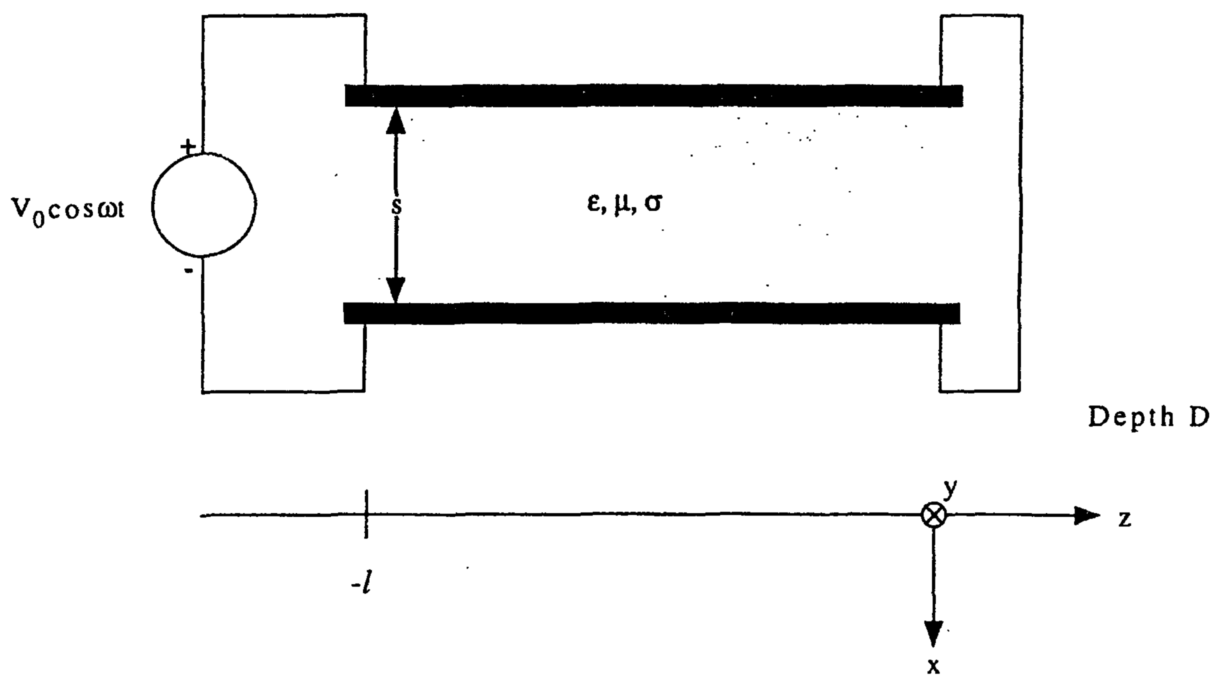

Figure 10: A lossy transmission line

A lossy transmission line is composed of perfectly conducting parallel plates enclosing a lossy medium with dielectric permittivity ϵ, magnetic permeability μ, and ohmic conductivity σ. The governing equations for the voltage $v(z,t)$ and current $i(z,t)$ along the transmission line are

$$\frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t} - Gv$$
$$\frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t}$$

Where C is the capacitance per unit length, G is the conductance per unit length, and L is the inductance per unit length. The transmission line is short circuited at $z = 0$ and is driven by a voltage source at $z = -l, v(z = -l, t) = V_0 \cos(\omega t)$.

(A) What are C, G, and L in terms of $\epsilon, \mu, \sigma, l, D$ and s?

Solution:

$$C = \frac{\epsilon d}{s}, L = \frac{\mu s}{d}, G = \frac{\sigma d}{s} \qquad \left(RC = \frac{\epsilon}{\sigma} \Rightarrow G = \frac{1}{R} = \frac{\sigma C}{\epsilon}\right)$$

(B) In the sinusoidal steady state the voltage and current can be written in the form

$$v(z,t) = Re\left[\hat{v}(z)e^{j\omega t}\right]$$
$$i(z,t) = Re\left[\hat{i}(z)e^{j\omega t}\right]$$

Find $\hat{v}(z)$ for this problem.

Solution:

$$\frac{d\hat{i}}{dz} = -(G + Cj\omega)\hat{v}$$

$$\frac{d\hat{v}}{dz} = -Lj\omega\hat{i} \Rightarrow \hat{i} = -\frac{1}{Lj\omega}\frac{d\hat{v}}{dz}$$

$$\frac{1}{Lj\omega}\frac{d^2\hat{v}}{dz^2} = +(G + Cj\omega)\hat{v}$$

$$\frac{d^2\hat{v}}{dz^2} = (GLj\omega - LC\omega^2)\hat{v}$$

$$\hat{v}(z) = Ae^{jpz} \Rightarrow -p^2 = GLj\omega - LC\omega^2$$
$$p = \pm\sqrt{LC\omega^2 - GLj\omega}$$
$$p = \pm p_0, p_0 = \sqrt{LC\omega^2 - GLj\omega}$$

$$\hat{v}(z) = A_1e^{jp_0z} + A_2e^{-jp_0z}$$

$$\hat{v}(z = 0) = 0 = A_1 + A_2$$

$$\hat{v}(z = -l) = V_0 = A_1e^{-jp_0l} + A_2e^{jp_0l} = A_1(e^{-jp_0l} - e^{jp_0l}) = -2jA_1 \sin p_0l$$

$$A_1 = -A_2 = -\frac{V_0}{2j\sin p_0l}$$

$$\hat{v}(z) = -\frac{V_0}{2j\sin p_0l}(e^{jp_0z} - e^{-jp_0z}) = -\frac{V_0\sin p_0z}{\sin p_0l}$$

(C) Find $\hat{i}(z)$ for this problem.

Solution:

$$\hat{i}(z) = -\frac{1}{Lj\omega}\frac{d\hat{v}}{dz} = -\frac{1}{Lj\omega}\left(-\frac{V_0p_0\cos p_0z}{\sin p_0l}\right) = \frac{V_0p_0}{Lj\omega}\frac{\cos p_0z}{\sin p_0l}$$

(D) Now, assuming $G = 0$ and neglecting fringing field effects, find the Poynting vector $\bar{S} = \bar{E} \times \bar{H}$ as a function of time and position z everywhere along the transmission line for $-l \leq z \leq 0$.

Solution:

$$G = 0 \Rightarrow p_0 = \omega\sqrt{LC} \quad \text{(real)}, v(z,t) = \text{Re}\hat{v}(z)e^{j\omega t} - \frac{V_0 \sin p_0 z}{\sin p_0 l}\cos\omega t$$

$$i(z,t) = Re\left[+\frac{V_0 p_0}{Lj\omega}\frac{\cos p_0 z}{\sin p_0 l}e^{j\omega t}\right] = \frac{V_0 p_0}{L\omega}\frac{\cos p_0 z}{\sin p_0 l}\sin\omega t = +V_0\sqrt{\frac{C}{L}}\frac{\cos p_0 z}{\sin p_0 l}\sin\omega t$$

$$E_x = \frac{v(z,t)}{s}, H_y = \frac{i(z,t)}{d} \Rightarrow \bar{S} = \bar{E}\times\bar{H} = E_x H_y \bar{i_z} = \frac{v(z,t)i(z,t)}{sd}\bar{i_z} = \frac{-V_0^2\sqrt{\frac{C}{L}}\sin p_0 z\cos p_0 z\sin\omega t\cos\omega t}{sd\sin^2 p_0 l}i$$

Solutions to Two More Problems

Problem 1

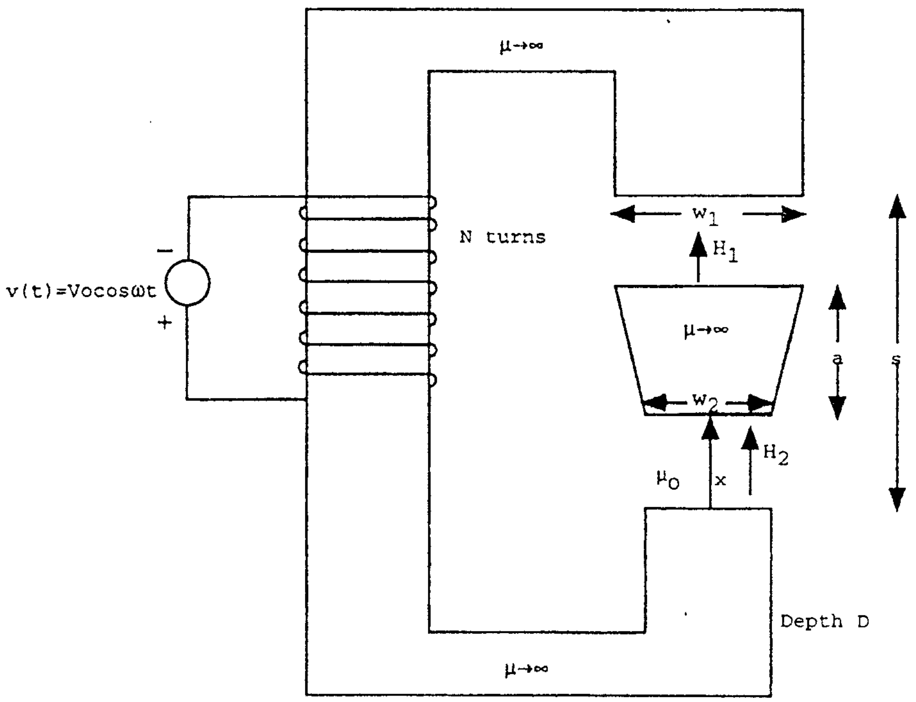

Figure 11: A magnetic circuit with a stationary yoke

A magnetic circuit has a stationary yoke with infinite magnetic permeability with a voltage source $v(t) = V_0 \cos \omega t$ exciting an N-turn perfectly conducting coil. In the air gap of the magnetic yoke of height s there is an infinitely magnetically permeable tapered wedge of height a ($a < s$) whose width decreases from w_1 to w_2. The bottom surface of the wedge is a distance x above the lower surface of the magnetic yoke. The system has depth D. Assume that both air gaps are sufficiently small to neglect fringing fields.

(A) What is the total magnetic flux $\lambda(t)$ linking the N-turn coil?

<u>Solution:</u>

$$v = V_0 \cos \omega t = \frac{d\lambda}{dt} \Rightarrow \lambda = \frac{V_0}{\omega} \sin \omega t$$

(B) What are the magnetic fields, $\bar{H}_1(t)$ and $\bar{H}_2(t)$, in the air gaps in terms of $\lambda(t), \mu_0$, and geometric parameters?

Solution:

$$\mu_0 H_1 w_1 D = \mu_0 H_2 w_2 D = \frac{\lambda}{N}$$

$$H_1 = \frac{\lambda}{N\mu_0 w_1 D}$$

$$H_2 = \frac{\lambda}{N\mu_0 w_2 D}$$

(C) What is the self-inductance $L(x)$ of the N turn coil as a function of the distance x, μ_0, and geometric parameters?

Solution:

$$\begin{aligned}\oint \bar{H}\cdot\overline{dl} = Ni &= H_1(s-a-x) + H_2 x \\ &= \frac{\lambda}{N\mu_0 D}\left(\frac{s-a-x}{w_1} + \frac{x}{w_2}\right)\end{aligned}$$

$$L(x) = \frac{\lambda}{i} = \frac{N^2\mu_0 D}{\left[\frac{s-a-x}{w_1} + \frac{x}{w_2}\right]}$$

(D) What is the x-directed force on the tapered wedge in terms of $\lambda(t)$, μ_0, and geometric parameters?

Solution:

$$\begin{aligned}f_x &= -\frac{1}{2}\lambda^2 \frac{d}{dx}\left[\frac{1}{L(x)}\right] \\ &= -\frac{\lambda^2}{2}\frac{d}{dx}\frac{\left[\frac{s-a-x}{w_1} + \frac{x}{w_2}\right]}{N^2\mu_0 D} \\ &= -\frac{\lambda^2}{2N^2\mu_0 D}\left[\frac{1}{w_2} - \frac{1}{w_1}\right]\end{aligned}$$

Problem 5

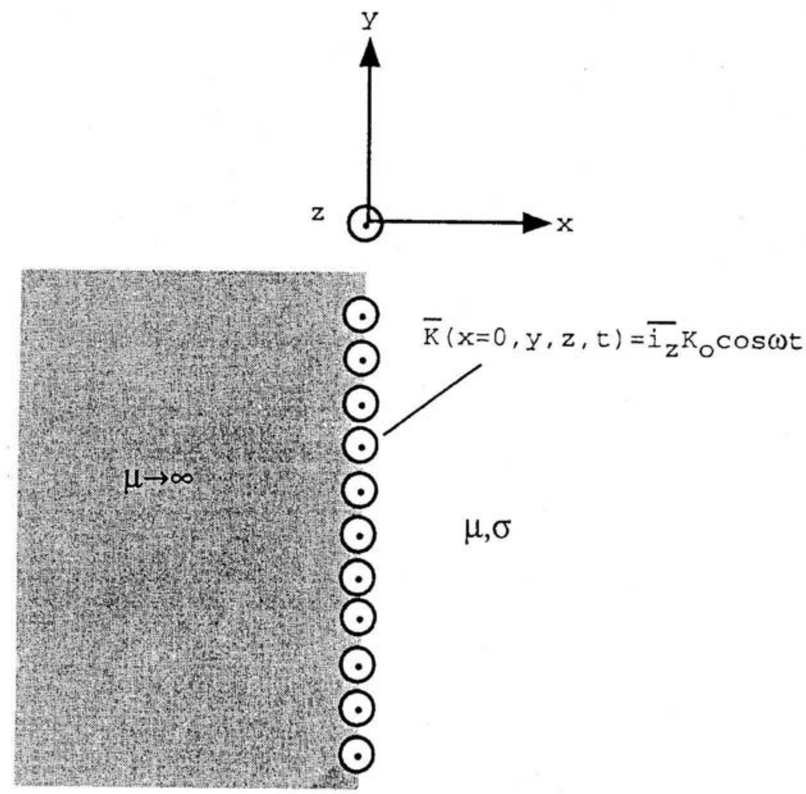

Figure 12: A current sheet of infinite extent

A z directed current sheet of infinite extent in the y and z directions is located at $x = 0$ and varies with time as

$$\bar{K}(x = 0, y, z, t) = \bar{i}_z K_0 \cos \omega t$$

This current sheet is located at the interface separating a material of infinite magnetic permeability ($\mu \to \infty$) for $-\infty < x < 0$ and a material of finite magnetic permeability μ and finite ohmic conductivity σ for $0 < x < \infty$. Note that because the current sheet has no variation with y or z, the magnetic field does not depend on the y or z coordinates.

(A) Find the magnitude and direction of the magnetic field $\bar{H}(x, t)$ everywhere.

Solution:

$$H_y(x, t) = \mathrm{Re}\hat{H}_y(x)e^{j\omega t}$$

$$\hat{H}_y(x = 0) = K_0 \Rightarrow \hat{H}_y(x) = K_0 e^{-\frac{(1+j)x}{\delta}}, \delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$\begin{aligned} H_y(x, t) &= \mathrm{Re}K_0 e^{-\frac{x}{\delta}} e^{-\frac{jx}{\delta}} e^{j\omega t} \\ &= K_0 e^{-\frac{x}{\delta}} \cos(\omega t - \frac{x}{\delta}) \qquad x > 0 \end{aligned}$$

$$H_y(x,t) = 0 \qquad x < 0$$

(B) Find the volume current density $\bar{J}(x,t)$ everywhere.

Solution:

$$\nabla \times \bar{H} = \bar{J} \Rightarrow \frac{\partial H_y}{\partial x} = J_z, J_z = 0, x < 0$$

$$J_z = \frac{\partial H_y}{\partial x} = \frac{K_0 e^{-\frac{x}{\delta}}}{\delta}\left[-\cos(\omega t - \frac{x}{\delta}) + \sin(\omega t - \frac{x}{\delta})\right]$$

(C) Find the power flow density, $\bar{S} = \bar{E} \times \bar{H}$, everywhere.

Solution:

$\underline{x > 0:}$

$$\begin{aligned}
\bar{S} = \bar{E} \times \bar{H} = \frac{\bar{J}}{\sigma} \times \bar{H} &= \frac{J_z}{\sigma} H_y \bar{i_z} \times \bar{i_y} \\
&= -\frac{J_z}{\sigma} H_y \bar{i_x} \\
&= -\frac{K_0^2 e^{-\frac{2x}{\delta}}}{\delta\sigma}\left[-\cos(\omega t - \frac{x}{\delta}) + \sin(\omega t - \frac{x}{\delta})\right]\cos(\omega t - \frac{x}{\delta}) \\
&= \frac{K_0^2}{\sigma\delta} e^{-\frac{2x}{\delta}} \cos(\omega t - \frac{x}{\delta})\left[\cos(\omega t - \frac{x}{\delta}) - \sin(\omega t - \frac{x}{\delta})\right]
\end{aligned}$$

$$\bar{S} = 0, x < 0$$

Final Exam 1995 Solutions

Problem 1

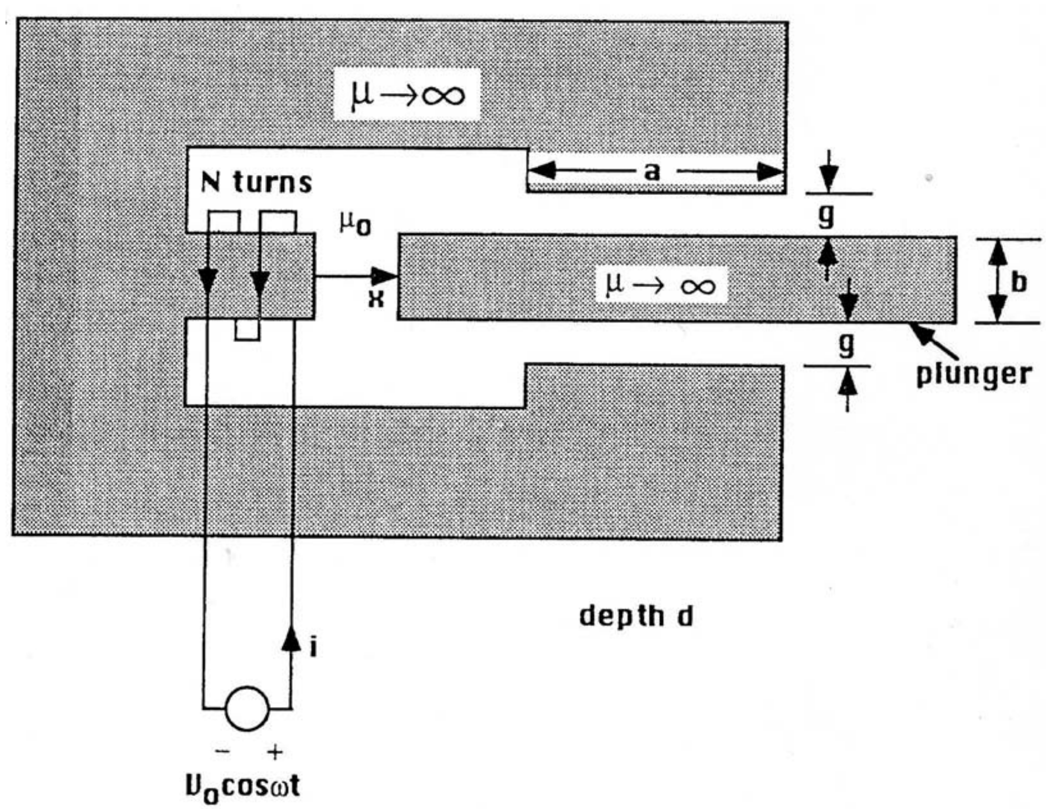

Figure 13: A magnetic circuit

The magnetic circuit shown above is modeled as being infinitely permeable except for the three thin air-gaps, where $\mu = \mu_0$. These thin gaps are narrow enough that fringing fields can be ignored. The N turn coil is driven by the voltage source $v(t) = V_0 \cos \omega t$.

(A) Determine the self-inductance $L(x)$ of the N turn coil.

Solution:

$$H_g g + H_x x = Ni$$

$$\mu_0 H_x bd = 2\mu_0 H_g ad \Rightarrow H_g = \frac{H_x b}{2a}$$

$$H_x \left[x + \frac{gb}{2a} \right] = Ni \Rightarrow H_x = \frac{Ni}{\left[x + \frac{gb}{2a} \right]}$$

$$V_0 \cos \omega t = \frac{d\lambda}{dt} \Rightarrow \lambda = \frac{V_0}{\omega} \sin \omega t = N\mu_0 H_x bd = \frac{\mu_0 bdN^2 i}{\left[x + \frac{gb}{2a} \right]}$$

$$L(x) = \frac{\lambda}{i} = \frac{\mu_0 b d N^2}{\left[x + \frac{gb}{2a}\right]}$$

(B) Find the total magnetic energy stored in the system as a function of time t in terms of V_0, ω, and given geometric and physical parameters.

<u>**Solution:**</u>

$$W_m = \frac{1}{2}\frac{\lambda^2}{L(x)} = \frac{1}{2}\frac{V_0^2 \sin^2 \omega t}{\omega^2 \mu_0 b d N^2}\left[x + \frac{gb}{2a}\right]$$

(C) Determine the magnetic force acting on the movable plunger in the x direction as a function of time t in terms of V_0, ω, and given geometric and physical parameters.

<u>**Solution:**</u>

$$f = -\frac{1}{2}\lambda^2 \frac{d}{dx}\left[\frac{1}{L(x)}\right] = -\frac{V_0^2 \sin^2 \omega t}{2\omega^2 \mu_0 b d N^2}$$

Problem 5

A sphere of magnetic material having radius R is to be magnetized by placing it in a source of uniform magnetic field intensity. The bulk of the sphere has a constant magnetic permeability μ with zero electrical conductivity, $\sigma = 0$. The magnetizable sphere is surrounded by a thin spherical shell of material with thickness $\Delta \ll R$ having electrical conductivity σ and magnetic permeability μ_0. The field source is switched on at $t = 0$ so that $\bar{H}_0(t) = H_0 u(t)\bar{i_z}$ where $u(t)$ is the unit step function in time.

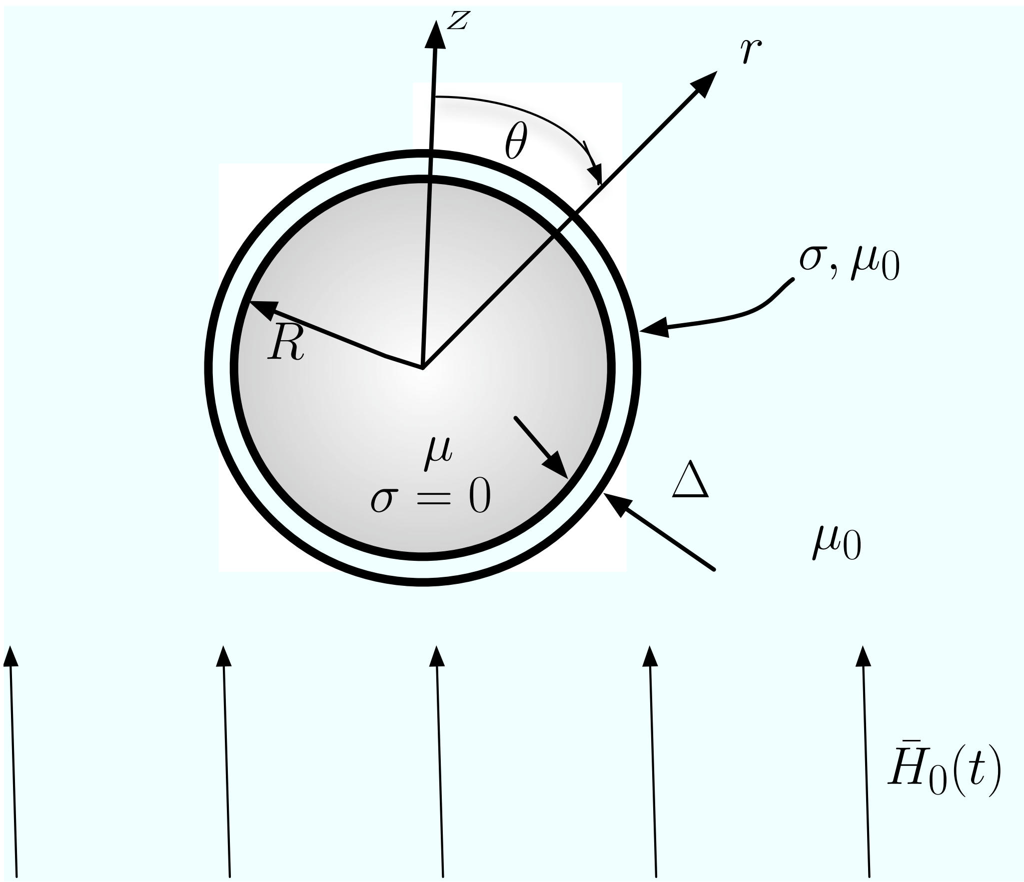

Figure 14: A sphere of magnetic material with a non-magnetic conducting coating (Image by MIT OpenCourseWare.)

(A) What is the magnetic field intensity $\bar{H}$ inside the magnetizable sphere for $r < R$ at $t = 0^+$ and at $t \to \infty$?

__Solution:__

$$\bar{H}(t = 0_+) = 0 \qquad r < R$$

$$t \to \infty : \bar{H} = -\nabla\chi = -\left[\frac{\partial\chi}{\partial r}\bar{i_r} + \frac{1}{r}\frac{\partial\chi}{\partial\theta}\bar{i_\theta}\right]$$

$$\nabla^2\chi = 0$$

$$\chi = \begin{cases} Ar\cos\theta & 0 < r < R \\ \left(Cr + \frac{D}{r^2}\right) cos\theta & r > R \end{cases}$$

$$\lim_{r\to\infty} \bar{H} = H_0 \bar{i_z} = H_0 \left[\cos\theta \bar{i_r} - \sin\theta \bar{i_\theta}\right]$$

$$\lim_{r\to\infty} \chi = -H_0 z = -H_0 r\cos\theta$$

$$C = -H_0$$

$$\bar{H} = \begin{cases} -A\left[\cos\theta\bar{i_r} - \sin\theta\bar{i_\theta}\right] & 0 < r < R \\ -\left[\left(-H_0 - \frac{2D}{r^3}\right)\cos\theta\bar{i_r} - \left(-H_0 + \frac{D}{r^3}\right)\sin\theta\bar{i_\theta}\right] & r > R \end{cases}$$

$$H_\theta(r = R_-) = H_\theta(r = R_+) \Rightarrow A = -H_0 + \frac{D}{R^3}$$

$$\mu H_r(r = R_-) = \mu_0 H_r(r = R_+) \Rightarrow -\mu A = -\mu_0\left(-H_0 - \frac{2D}{R^3}\right)$$

$$-\frac{\mu}{\mu_0}A = H_0 + \frac{2D}{R^3}$$

$$A = -H_0 + \frac{D}{R^3}$$

$$A = -\frac{3H_0}{2 + \frac{\mu}{\mu_0}}$$

$$\bar{H}(r < R, t \to \infty) = \frac{3H_0}{2 + \frac{\mu}{\mu_0}}\bar{i_z}$$

(B) The radial component of Faraday's law for this problem is:

$$\nabla \times \overline{E} = -\frac{\partial\overline{B}}{\partial t} \to \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\phi) = -\frac{\partial B_r}{\partial t}$$

Because $\Delta \ll R$, the current flow in the conducting spherical shell can be modeled as a surface current, $K_\phi(r = R)$. What is the approximate boundary condition at $r = R$ relating the tangential (θ) component of $\bar{H}$ on either side of the spherical shell to the perpendicular (radial) component of $\bar{B}$?

<u>Solution:</u>

$$\nabla \times \bar{E} = -\frac{\partial\bar{B}}{\partial t} \Rightarrow \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\phi) = -\frac{\partial B_r}{\partial t}$$

In spherical shell:

$$J_\phi = \sigma E_\phi = \frac{K_\phi}{\Delta} = \frac{1}{\Delta}\left[H_\theta(r = R_+) - H_\theta(r = R_-)\right]$$

At $r = R$:

$$\frac{1}{\sigma\Delta R\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(H_\theta(r = R_+) - H_\theta(r = R_-)\right)\right] = -\frac{\partial B_r}{\partial t}$$

(C) What is the approximate magnetic diffusion time τ_m for this configuration?

Solution:

$$H_\theta(r = R_+) - H_\theta(r = R_-) = \left(-H_0 + \frac{D}{R^3}\right)\sin\theta - A\sin\theta$$

$$\mu H_r(r = R_-) = \mu_0 H_r(r = R_+) \Rightarrow -\mu A = -\mu_0\left(-H_0 - \frac{2D}{R^3}\right) - \frac{1}{2}\left(\frac{\mu}{\mu_0}A + H_0\right) = \frac{D}{R^3}$$

$$H_\theta(r = R_+) - H_\theta(r = R_-) = \sin\theta\left[-H_0 - A + \frac{D}{R^3}\right] = \sin\theta\left[-H_0 - A - \frac{1}{2}\frac{\mu}{\mu_0}A - \frac{H_0}{2}\right]$$

$$\frac{1}{\sigma\Delta R\sin\theta}\frac{d}{d\theta}\left[\sin^2\theta\left(-\frac{3H_0}{2} - A\left(\frac{1}{2}\frac{\mu}{\mu_0} + 1\right)\right)\right] = \mu\frac{\partial A}{\partial t}\cos\theta$$

$$\frac{2}{\sigma\Delta R\mu}\left[-\frac{3H_0}{2} - A\left(\frac{1}{2}\frac{\mu}{\mu_0} + 1\right)\right] = \frac{\partial A}{\partial t} \Rightarrow \frac{dA}{dt} + \frac{2A}{\sigma\Delta R\mu}\left(\frac{1}{2}\frac{\mu}{\mu_0} + 1\right) = -\frac{3H_0}{\sigma\Delta R\mu}$$

$$\tau_m = \frac{\sigma\Delta R\mu}{\left(\frac{\mu}{\mu_0} + 2\right)}$$

Final Exam 2000 Solutions

Problem 1

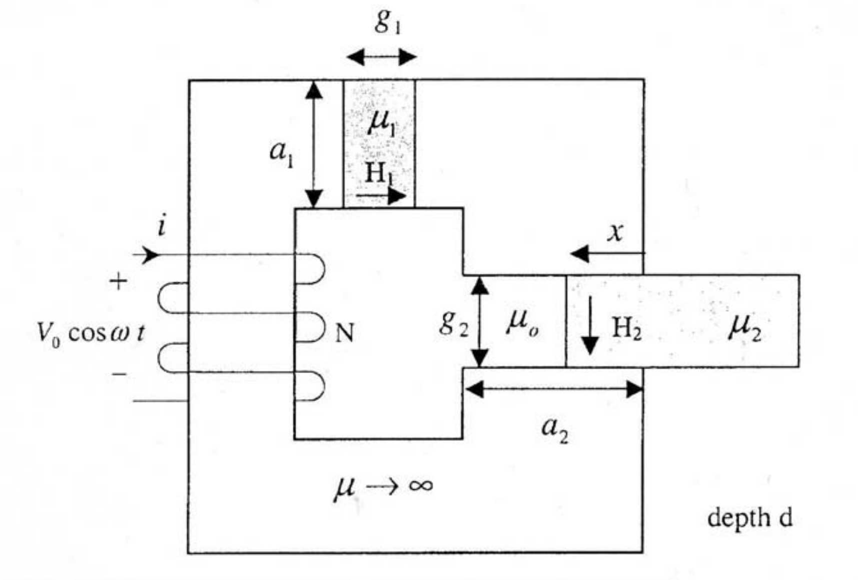

Figure 15: A magnetic circuit with a gap

The magnetic circuit shown above is modeled as being infinitely permeable except for the gap g_1 of material with magnetic permeability μ_1, and the free space gap partially filled with material with magnetic permeability μ_2. The two gaps are sufficiently narrow that fringing fields are negligible. The N turn coil is driven by the voltage source $v(t) = V_0 \cos\omega t$.

(A) What is the magnetic flux λ through the N turn coil in terms of the terminal voltage?

Solution:

$$v = V_0 \cos\omega t = \frac{d\lambda}{dt} \Rightarrow \lambda = \frac{V_0}{\omega}\sin\omega t$$

(B) What are the magnetic fields H_1 and H_2 in the two gaps in terms of the magnetic flux, λ, magnetic permeabilities, and geometric factors?

Solution:

$$\Phi = \frac{\lambda}{N} = \mu_1 H_1 a_1 d = H_2 d\left(\mu_2 x + \mu_0(a_2 - x)\right)$$

$$H_1 = \frac{\lambda}{a} N\mu_1 a_1 d, H_2 = \frac{\lambda}{Nd\left[\mu_2 x + \mu_0(a_2 - x)\right]}$$

(C) What is the coil current i?

__Solution:__

$$H_1 g_1 + H_2 g_2 = Ni = \frac{\lambda g_1}{N\mu_1 a_1 d} + \frac{\lambda g_2}{Nd\left[\mu_2 x + \mu_0(a_2 - x)\right]}$$

$$i = \frac{\lambda}{N^2 d}\left[\frac{g_1}{\mu_1 a_1} + \frac{g_2}{\left[\mu_2 x + \mu_0(a_2 - x)\right]}\right]$$

(D) What is the self-inductance $L(x)$ of the N turn coil where x is the penetration distance of the material with magnetic permeability μ_2 into the free space gap?

__Solution:__

$$L(x) = \frac{\lambda}{i} = \frac{N^2 d}{\left[\frac{g_1}{\mu_1 a_1} + \frac{g_2}{\left[\mu_2 x + \mu_0(a_2 - x)\right]}\right]}$$

(E) What is the magnetic stored energy?

__Solution:__

$$W_m = \frac{1}{2}L(x)i^2 = \frac{1}{2}\frac{\lambda^2}{L(x)}$$

(F) Determine the magnitude and direction of the magnetic force on the movable slab with magnetic permeability μ_2.

__Solution:__

$$f_x = \frac{1}{2}i^2\frac{dL}{dx} = -\frac{\lambda^2}{2}\frac{d}{dx}\left(\frac{1}{L(x)}\right)$$

$$\frac{1}{L(x)} = \frac{\left[\frac{g_1}{\mu_1 a_1} + \frac{g_2}{\left[\mu_2 x + \mu_0(a_2 - x)\right]}\right]}{N^2 d}$$

$$f_x = -\frac{\lambda^2}{2N^2 d}g_2\frac{-(\mu_2 - \mu_0)}{\left[\mu_2 x + \mu_0(a_2 - x)\right]^2} = \frac{\lambda^2 g_2}{2N^2 d}\frac{(\mu_2 - \mu_0)}{\left[\mu_2 x + \mu_0(a_2 - x)\right]^2}$$

Problem 2

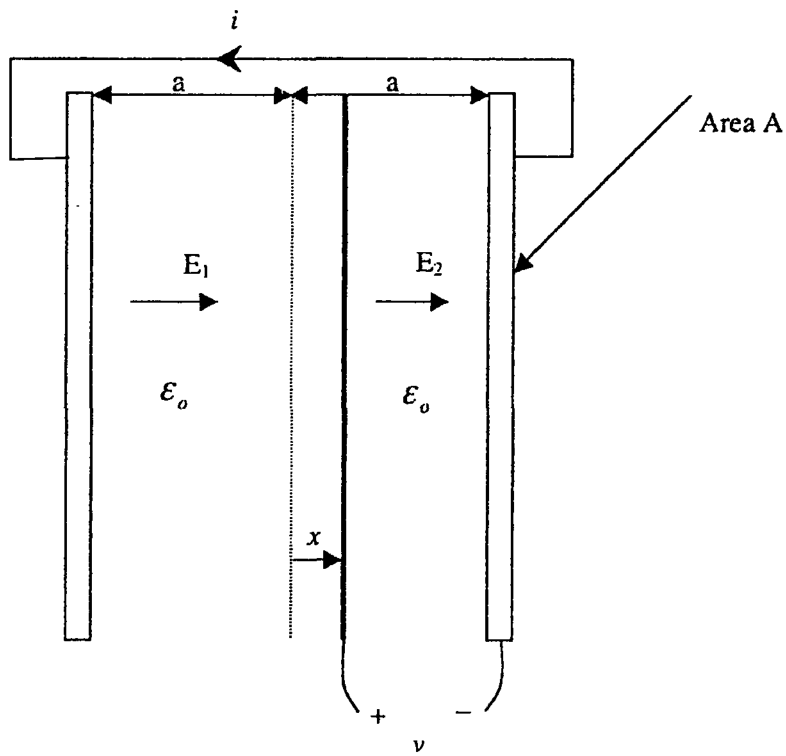

Figure 16: Short circuited parallel plate electrodes

Two parallel plate electrodes of area A in free space are a distance $2a$ apart and are short circuited together. A third electrode at potential v with respect to the other two electrodes and with negligible thickness is placed at a distance x to the right of the midpoint position of the two short circuited electrodes.

(A) Find the electric fields E_1 and E_2 on either side of the middle electrode. Neglect fringing field effects.

Solution:

$$E_1 = -\frac{v}{a+x}, E_2 = \frac{v}{a-x}$$

(B) What is the total charge on the middle electrode?

Solution:

$$q_{\text{mid}} = \epsilon_0(E_2 - E_1)A = \epsilon_0 v A\left(\frac{1}{a-x} + \frac{1}{a+x}\right) = \frac{2\epsilon_0 v A a}{(a^2 - x^2)}$$

(C) What is the capacitance of the middle electrode with respect to the short circuited electrodes?

Solution:

$$C = \frac{q_{\text{mid}}}{v} = \frac{2\epsilon_0 a A}{a^2 - x^2}$$

(D) If the voltage $v = v(t)$ and position $x = x(t)$ are functions of time, what is the current i flowing in the short circuit?

Solution:

$$i = \epsilon_0 A \frac{dE_1}{dt} = -\epsilon_0 A \left(\frac{1}{a+x}\frac{dv}{dt} - \frac{v}{(a+x)^2}\frac{dx}{dt} \right) = -\frac{\epsilon_0 A}{(a+x)} \left(\frac{dv}{dt} - \frac{v}{(a+x)}\frac{dx}{dt} \right)$$

(E) What is the electric force on the middle electrode as a function of x, v, ϵ_o, and geometric parameters a and A?

Solution:

$$f_x = \frac{1}{2}v^2 \frac{dC}{dx} = \frac{1}{\cancel{2}}v^2(\cancel{2}\epsilon_0 a A)\left(-\frac{1(-2x)}{(a^2-x^2)^2}\right) = \frac{2\epsilon_0 a A x v^2}{(a^2-x^2)^2}$$

Problem 4

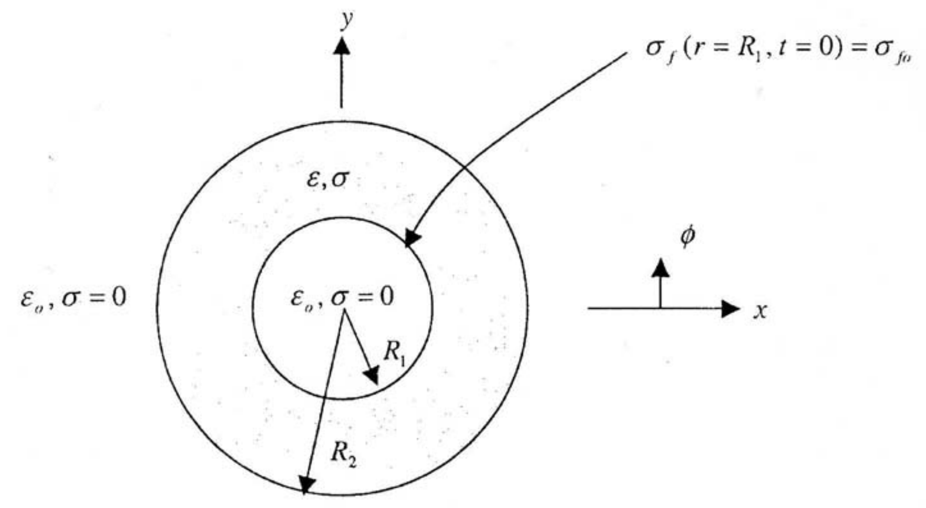

Figure 17: An infinitely long surface charged cylinder

An infinitely long cylinder with dielectric permittivity ϵ and ohmic conductivity σ has outer radius R_2 and free space hole of radius R_1. The cylinder is surrounded by free space for $r > R_2$. At time $t = 0$ a uniform surface charge distribution is placed at $r = R_1$ so that $\sigma_f(r = R_1, t = 0) = \sigma_{f0}$. At time $t = 0$ the free surface charge distribution at $r = R_2$ is zero.

(A) What is the electric field in the regions $r < R_1$, $R_1 < r < R_2$ and $r > R_2$ at time $t = 0$?

<u>**Solution:**</u> At $t = 0$:

$$E_r = \begin{cases} 0 & r < R_1 \\ \frac{\sigma_{f0}\cancel{2\pi} R_1}{\cancel{2\pi} r\epsilon} & R_1 < r < R_2 \\ \frac{\sigma_{f0}\cancel{2\pi} R_1}{\cancel{2\pi} r\epsilon_0} & r > R_2 \end{cases}$$

(B) Find the electric field in the regions $r < R_1$, $R_1 < r < R_2$, and $r > R_2$ as a function of time.

<u>**Solution:**</u>

$$\sigma E_r(r = R_{1+}) + \epsilon\frac{\partial E_r(r = R_{1+})}{\partial t} = 0$$

$$E_r(r = R_{1+}, t) = E_r(r = R_{1+}, t = 0)e^{-\frac{t}{\tau}}; \tau = \frac{\epsilon}{\sigma}$$

$$\sigma_f(r = R_1) = \epsilon E_r(r = R_{1+}, t) = \epsilon E_r(r = R_{1+}, t = 0)e^{-\frac{t}{\tau}} = \sigma_{f0}e^{-\frac{t}{\tau}}$$

$$E_r(r, t) = \begin{cases} 0 & r < R_1 \\ \frac{\sigma_{f0}R_1}{\epsilon r}e^{-\frac{t}{\tau}} & R_1 < r < R_2 \\ \frac{\sigma_{f0}R_1}{\epsilon_0 r} & r > R_2 \end{cases}$$

(C) Find the free surface charge distributions as a function of time at $r = R_1$ and $r = R_2$.

Solution:

$$\sigma_f(r = R_1, t) = \sigma_{f_0} e^{-\frac{t}{\tau}}$$

$$-\sigma E_r(r = R_{2-}, t) + \frac{\partial \sigma_f(r = R_2, t)}{\partial t} = 0$$

$$\frac{\partial \sigma_f(r = R_2, t)}{\partial t} = +\sigma E_r(r = R_{2-}, t) = +\frac{\sigma}{\epsilon}\frac{\sigma_{f_0}}{R_2} R_1 e^{-\frac{t}{\tau}}$$

$$\begin{aligned}
\sigma_f(r = R_2, t) &= +\frac{\sigma}{\epsilon}\frac{\sigma_{f_0} R_1}{R_2}(-\tau) e^{\frac{t}{\tau}} + C \\
&= \frac{\cancel{\sigma}}{\cancel{\epsilon}}\frac{\sigma_{f_0} R_1}{R_2}\left(-\frac{\cancel{\epsilon}}{\cancel{\sigma}}\right) e^{-\frac{t}{\tau}} + C \\
&= \frac{-\sigma_{f_0} R_1}{R_2} e^{-\frac{t}{\tau}} + C
\end{aligned}$$

$$\sigma_f(r = R_2, t = 0) = 0 = \frac{-\sigma_{f_0} R_1}{R_2} + C = 0 \Rightarrow C = \frac{\sigma_{f_0} R_1}{R_2}$$

$$\sigma_f(r = R_2, t) = \frac{\sigma_{f_0} R_1}{R_2}\left(1 - e^{-t/\tau}\right)$$

Another Way:

$$\begin{aligned}
\sigma_f(r = R_2, t) &= \epsilon_0 E_r(r = R_{2+}, t) - \epsilon E_r(r = R_{2-}, t) \\
&= \frac{\sigma_{f_0} R_1}{R_2} - \frac{\sigma_{f_0} R_1}{R_2} e^{-\frac{t}{\tau}} \\
&= \frac{\sigma_{f_0} R_1}{R_2}\left(1 - e^{-\frac{t}{\tau}}\right)
\end{aligned}$$

Another way:

$$\sigma_f(r = R_1, t)\cancel{2\pi} R_1 + \sigma_f(r = R_2, t)\cancel{2\pi} R_2 = \sigma_{f_0}(\cancel{2\pi} R_1)$$

$$\begin{aligned}
\sigma_f(r = R_2, t) &= \frac{\sigma_{f_0} R_1}{R_2} - \sigma_f(r = R_1, t)\frac{R_1}{R_2} \\
&= \frac{\sigma_{f_0} R_1}{R_2}\left(1 - e^{-\frac{t}{\tau}}\right)
\end{aligned}$$

Problem 5

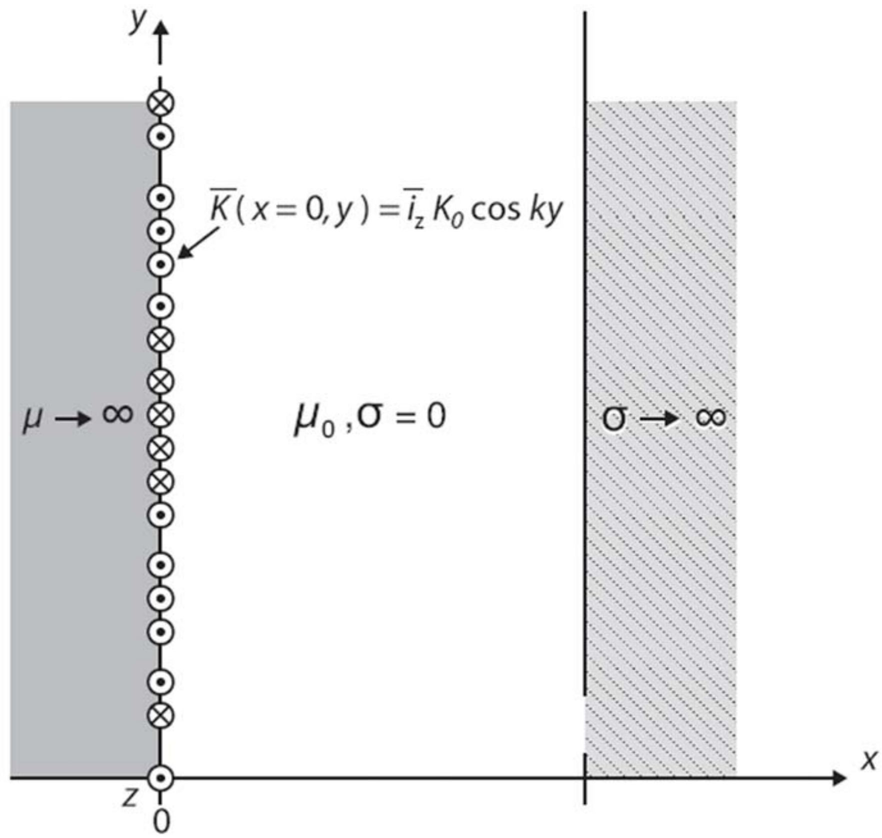

Figure 18: A surface current sheet at $x = 0$ (Image by MIT OpenCourseWare.)

A z directed surface current sheet of infinite extent in the y and z directions is located at $x = 0$ and varies with coordinate y as $\bar{K}(x = 0, y) = \bar{i_z} K_0 \cos ky$. This current sheet is located at the $x = 0$ interface separating a material of infinite magnetic permeability ($\mu \to \infty$) for $x < 0$ and free space for $0 < x < s$. At $x = s$ there is another material of infinite extent for $x > s$ with infinite ohmic conductivity ($\sigma \to \infty$). There is no variation with the z coordinates and free space for $0 < x < s$ is perfectly insulating ($\sigma = 0$).

(A) What are the boundary conditions on the magnetic field $\bar{H}(x, y)$ at $x = 0$ and $x = s$?

Solution:

$$H_y(x = 0_+) = K_o \cos ky$$

$$H_x(x = s_-) = 0$$

(B) Find the magnetic field $\bar{H}(x, y)$ everywhere.

Solution:

$$\chi(x, y) = \sin ky(Ae^{-kx} + Ce^{+kx}) \qquad 0 < x < s$$

$$\bar{H} = -\nabla\chi = \begin{cases} 0 & x < 0 \\ 0 & x > s \\ -\left[-kAe^{-kx} + kCe^{+kx}\right]\sin ky\bar{i_x} - k\cos ky\left[Ae^{kx} + Ce^{+kx}\right]\bar{i_y} & 0 < x < s \end{cases}$$

$$H_x(x = s_-) = 0 \Rightarrow -\cancel{k}Ae^{-ks} + \cancel{k}Ce^{ks} = 0$$

$$H_y(x = 0_+) = K_0\cancel{\cos ky} = -k\cancel{\cos ky}(A + C)$$

$$A + C = -\frac{K_0}{k}$$

$$A = Ce^{2ks} \Rightarrow C(1 + e^{2ks}) = -\frac{K_0}{k}$$

$$C = -\frac{\frac{K_0}{k}}{(1 + e^{2ks})}$$

$$A = -\frac{K_0e^{2ks}}{k(1 + e^{2ks})}$$

$$\bar{H} = \begin{cases} 0 & x < 0 \\ 0 & x > s \\ -\frac{K_0}{(1+e^{2ks})}\left[\sin(ky)\left(e^{-kx}e^{2ks} - e^{kx}\right)\bar{i_x} - \cos(ky)\left(e^{-kx}e^{2ks} + e^{kx}\right)\bar{i_y}\right] & 0 < x < s \end{cases}$$

$0 < x < s$

$$\begin{aligned} \bar{H} &= -\frac{2K_0e^{ks}}{(1 + e^{2ks})}\left[\sin(ky)\left(-\sinh(k(x - s))\right)\bar{i_x} - \cos(ky)\cosh(k(x - s))\bar{i_y}\right] \\ &= \frac{K_0}{\cosh(ks)}\left[\sin(ky)\left(\sinh(k(x - s))\right)\bar{i_x} + \cos(ky)\cosh(k(x - s))\bar{i_y}\right] \end{aligned}$$

Check:

$$H_x(x = s) = 0$$

$$H_y(x = 0) = K_0\cos(ky)$$

(C) What is the surface current on the $x = s$ surface?

Solution:

$$K_z(x = s) = -H_y(x = s) = \frac{K_0\cos(ky)}{\cosh(ks)}$$