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6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

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6.641 — Electromagnetic Fields, Forces, and Motion	Spring 2008
Quiz 1 - Solutions	
Prof. Markus Zahn	MIT OpenCourseWare

Problem 1

A

Question: Find a suitable image current to find the magnetic field for $z > 0$. Does the direction of the image current surprise you?
Solution:

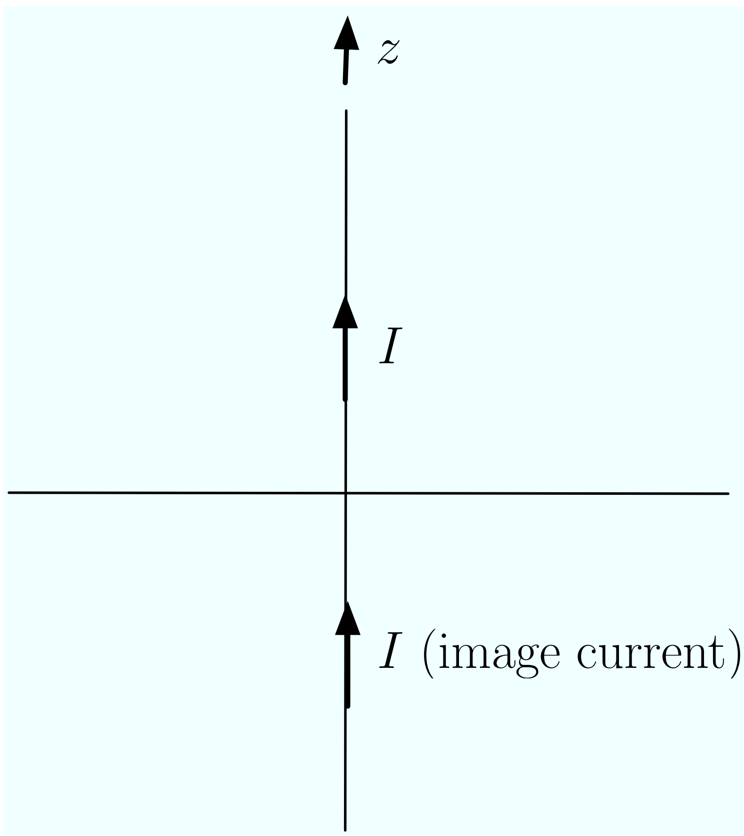

Figure 1: Figure showing image line. (Image by MIT OpenCourseWare).

B

Question: What is the magnetic field magnitude and direction for $z > 0$?
Solution:

$$H_\phi = \frac{I}{2\pi r} \qquad \text{for } z > 0$$

C

Question: What is the surface current magnitude and direction on the $z = 0$ surface of the conducting plane?

Solution:

$$\overline{n} \times \overline{H}(z = 0_+) = \overline{i}_z \times H_\phi(z = 0_+)\overline{i}_\phi = -H_\phi \overline{i}_r = \overline{K}$$

$$K_r = -H_\phi(z = 0_+) = -\frac{I}{2\pi r}$$

Problem 2

A

Question: What is the electric field for $a \leq r \leq b$?
Solution:

$$\nabla \cdot \overline{J} = \nabla \cdot \left[\sigma(r)\overline{E}\right] = 0 \qquad \left(\overline{E} = E_r(r)\overline{i}_r\right)$$

$$\nabla \cdot \left[\sigma(r)\overline{E}\right] = \frac{1}{r}\frac{\partial}{\partial r}(r\sigma(r)E_r(r)) = 0$$

$$\sigma(r) = \frac{\sigma_0 r}{a}$$

$$r\sigma(r)E_r(r) = C(\text{Constant}) = \frac{r^2\sigma_0}{a}E_r(r)$$

$$E_r(r) = \frac{Ca}{\sigma_0 r^2}$$

$$v = \int_{r=a}^{b} E_r(r)dr = \int_{r=a}^{b} \frac{Ca}{\sigma_0 r^2}dr = -\frac{Ca}{\sigma_0 r}\bigg|_{r=a}^{b} = -\frac{Ca}{\sigma_0}\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$C = \frac{\sigma_0 v}{1 - \frac{a}{b}} \Rightarrow E_r(r) = \frac{\cancel{\sigma_0} va}{\cancel{\sigma_0} r^2\left(1 - \frac{a}{b}\right)} = \frac{va}{r^2\left(1 - \frac{a}{b}\right)}$$

B

Question: What are the surface charge densities at $r = a$ and $r = b$?
Solution:

$$\sigma_s(r = a) = \varepsilon E_r(r = a) = \frac{\varepsilon va}{a^2\left(1 - \frac{a}{b}\right)} = \frac{\varepsilon v}{a\left(1 - \frac{a}{b}\right)}$$

$$\sigma_s(r = b) = -\varepsilon E_r(r = b) = -\frac{\varepsilon va}{b^2\left(1 - \frac{a}{b}\right)}$$

C

Question: **What is the volume charge density for $a \leq r \leq b$?**

Solution:

$$\rho = \varepsilon \nabla \cdot \overline{E} = \frac{\varepsilon}{r}\frac{\partial}{\partial r}(rE_r) = \frac{\varepsilon}{r}\frac{\partial}{\partial r}\left(\frac{va}{r\left(1-\frac{a}{b}\right)}\right)$$
$$= -\frac{\varepsilon}{r^3}\frac{va}{\left(1-\frac{a}{b}\right)}$$

D

Question: **What is the total charge in the system?**

Solution:

$$L\int_a^b \rho 2\pi r dr = -\frac{2\pi\varepsilon vaL}{\left(1-\frac{a}{b}\right)}\int_a^b \frac{1}{r^2}dr = \frac{2\pi\varepsilon vaL}{\left(1-\frac{a}{b}\right)}\frac{1}{r}\bigg|_a^b$$
$$= \frac{2\pi\varepsilon vaL}{\left(1-\frac{a}{b}\right)}\left(\frac{1}{b}-\frac{1}{a}\right)$$
$$= -2\pi\varepsilon vL$$

$$Q_T = \left[2\pi a\sigma_s(r=a) + 2\pi b\sigma_s(r=b) + \int_a^b \rho 2\pi r dr\right]L$$
$$= \frac{2\pi a\varepsilon v}{\left(1-\frac{a}{b}\right)}\left[\frac{1}{a}-\frac{1}{b}+\frac{1}{b}-\frac{1}{a}\right]L$$
$$= 0$$

E

Question: **What is the resistance between the cylindrical electrodes?**

Solution:

$$i = \sigma(r)E_r(r)2\pi rL = \frac{\sigma_0 \cancel{r}}{\cancel{a}}2\pi\cancel{r}L\frac{v\cancel{a}}{\cancel{r^2}\left(1-\frac{a}{b}\right)}$$
$$= \frac{\sigma_0 2\pi Lv}{\left(1-\frac{a}{b}\right)}$$

$$R = \frac{v}{i} = \frac{\left(1-\frac{a}{b}\right)}{\sigma_0 2\pi L}$$

Problem 3

A

Question: There is no volume charge for $0 < r < R$ and $r > R$ and $\Phi(r = \infty, \theta) = 0$. Laplace's equation for the scalar electric potential in spherical coordinates is:

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{d\Phi}{dr}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

Guess a solution to Laplace's equation of the form $\Phi(r,\theta) = Ar^p\cos\theta$ and find all allowed values of p.

Solution:

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

$$\Phi(r,\theta) = Ar^p\cos\theta$$

$$\begin{aligned}
\nabla^2\Phi &= \frac{1}{\cancel{r^2}}\frac{\partial}{\partial r}\left(r^2 Apr^{p-1}\cos\theta\right) + \frac{1}{\cancel{r^2}\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left(-Ar^p\sin\theta\right)\right) = 0 \\
0 &= Ap\cos\theta\frac{\partial}{\partial r}\left(r^{p+1}\right) - \frac{1}{\sin\theta}Ar^p\frac{\partial}{\partial\theta}\left(\sin^2\theta\right) \\
&= Ar^p\cos\theta\left(p\left(p+1\right)\right) - \frac{Ar^p}{\cancel{\sin\theta}}2\cancel{\sin\theta}\cos\theta \\
&= Ar^p\cos\theta\left[p(p+1) - 2\right] = 0
\end{aligned}$$

$$p^2 + p - 2 = (p+2)(p-1) = 0 \Rightarrow p = 1, p = -2$$

$$\Phi_1(r,\theta) = Ar\cos\theta, \Phi_2(r,\theta) = \frac{A\cos\theta}{r^2}$$

B

Question: Which of your scalar electric potential solutions in part (a) are finite at $r = 0$?

Solution:

$$\Phi_1(r,\theta) = Ar\cos\theta$$

C

Question: Which of your solutions in part (a) have zero potential at $r = \infty$?

Solution:

$$\Phi_2(r,\theta) = \frac{A\cos\theta}{r^2}$$

D

Question: Using the results of parts (b) and (c) find the scalar electric potential solutions for $0 \le r \le R$ and $r \ge R$ that satisfy the boundary condition $\Phi(r = R, \theta) = V_0 \cos\theta$.

Solution:

$$\Phi(r, \theta) = \begin{cases} Ar\cos\theta & 0 \le r \le R \\ \frac{B}{r^2}\cos\theta & r \ge R \end{cases}$$

$$\Phi(r = R, \theta) = V_0 \cos\theta = AR\cos\theta = \frac{B}{R^2}\cos\theta$$

$$A = \frac{V_0}{R}, B = V_0 R^2$$

$$\Phi(r, \theta) = \begin{cases} \frac{V_0 r}{R}\cos\theta & 0 \le r \le R \\ \frac{V_0 R^2}{r^2}\cos\theta & r \ge R \end{cases}$$

E

Question: Find the electric field in the regions $0 \le r < R$ and $r > R$.

Hint:

$$\overline{E} = -\nabla\Phi = -\left[\frac{\partial\Phi}{\partial r}\overline{i_r} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\overline{i_\theta}\right]$$

Solution:

$$\overline{E} = -\left[\frac{\partial\Phi}{\partial r}\overline{i_r} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\overline{i_\theta}\right]$$

$0 \le r < R$

$$\overline{E} = -\frac{V_0}{R}\left[\cos\theta\overline{i_r} - \sin\theta\overline{i_\theta}\right]$$

$r > R$

$$\overline{E} = -V_0 R^2\left[-\frac{2}{r^3}\cos\theta\overline{i_r} - \frac{\sin\theta}{r^3}\overline{i_\theta}\right]$$

$$= \frac{V_0 R^2}{r^3}\left(2\cos\theta\overline{i_r} + \sin\theta\overline{i_\theta}\right)$$

F

Question: What is the surface charge distribution on the $r = R$ interface?

Solution:

$$\sigma_s(r = R, \theta) = \varepsilon_0 E_r(r = R_+, \theta) - \varepsilon E_r(r = R_-, \theta)$$

$$= \frac{\varepsilon_0 V_0}{R} 2\cos\theta + \frac{\varepsilon V_0}{R}\cos\theta$$

$$= \frac{V_0}{R}(\varepsilon + 2\varepsilon_0)\cos\theta$$