6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

I. Governing Equations

Thin elastic rod with axis in the x_1-direction and uniform cross section of area A
(a) the rod; (b) force and tractions applied to an element of length Δx_1 centered
at x_1 with displacement $\delta(x_1, t)$.
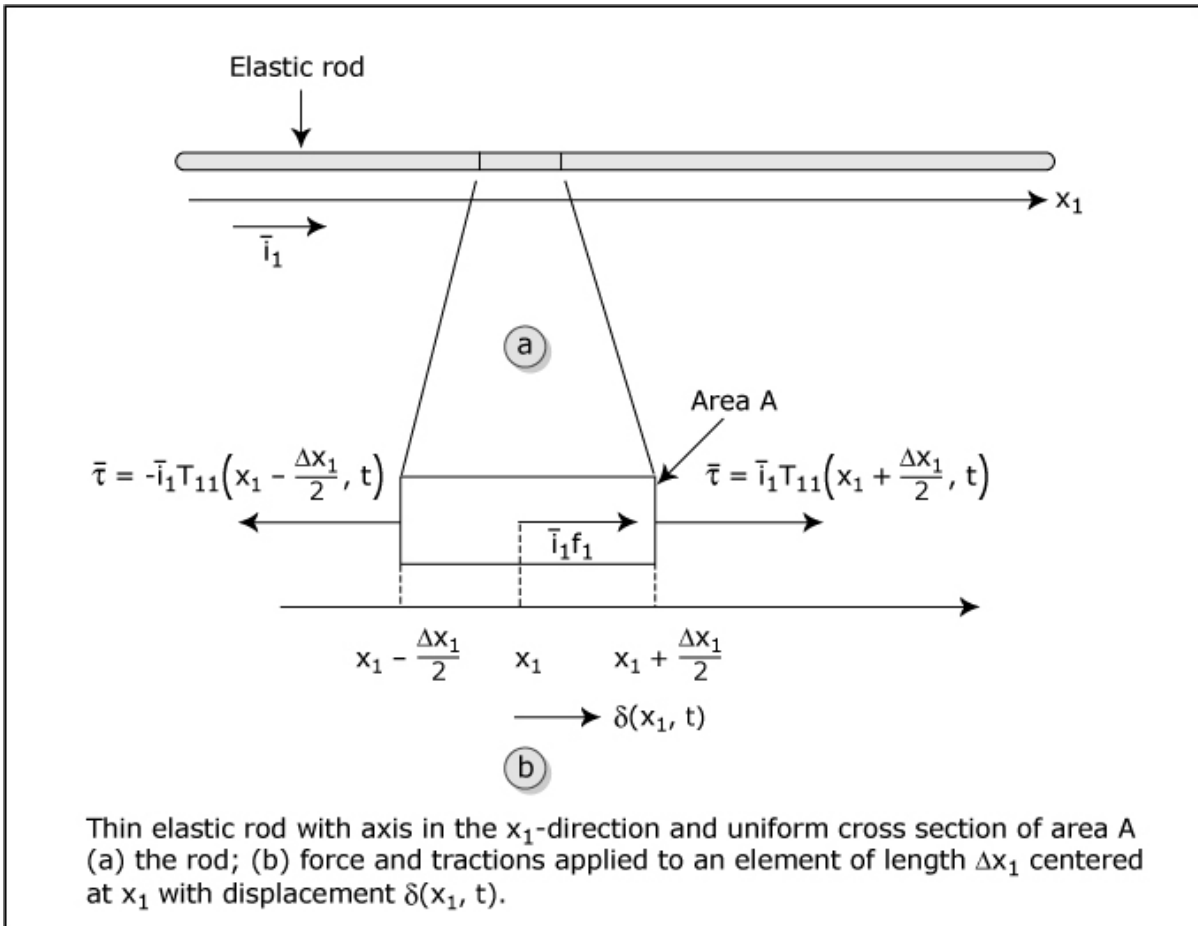
$$A \, \Delta x_1 \, \rho \, \frac{\partial^2 \delta}{\partial t^2} = \left[T_{11}\left(x_1 + \frac{\Delta x_1}{2}\right) - T_{11}\left(x_1 - \frac{\Delta x_1}{2}\right) \right] A + F \, A \, \Delta x_1$$

$$\rho \, \frac{\partial^2 \delta}{\partial t^2} = \frac{\partial \, T_{11}}{\partial x_1} + F \qquad (F \text{ is body force density})$$

$$T_{11} = E \, \frac{\partial \, \delta}{\partial x_1} \qquad \text{Hooke's Law (stress-strain relation)}$$

$\uparrow$

Young's modulus (modulus of elasticity)

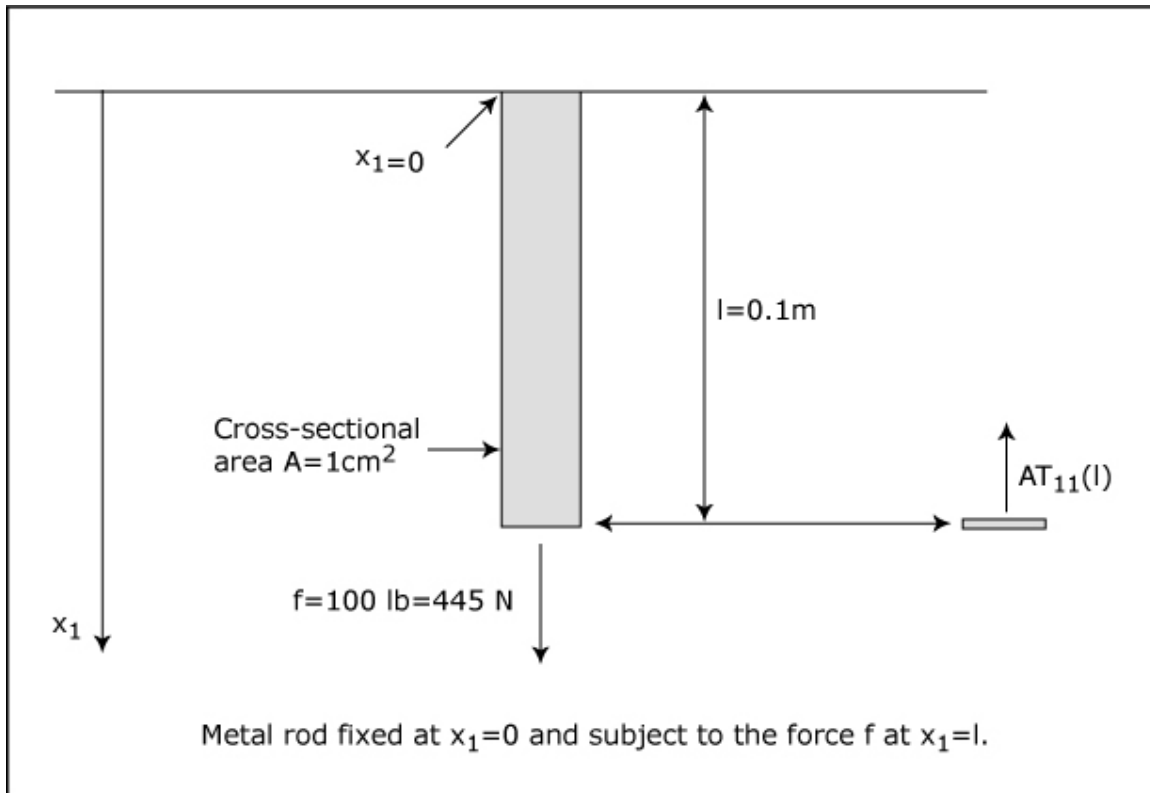$$\rho \frac{\partial^2 \delta}{\partial t^2} = E \frac{\partial^2 \delta}{\partial x_1^2} + F$$

6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn

Lecture 16
Page 1 of 17

Modulus of Elasticity E and Density ρ for Representative Materials

Material	E-units of 10^{11} N/m^2	ρ-units of 10^3 kg/m^3	v_p-units+ m/sec
Aluminum (pure and alloy)	0.68-0.79	2.66-2.89	5100
Brass (60-70% Cu, 40-30% Zn)	1.0-1.1	8.36-8.51	3500
Copper	1.17-1.24	8.95-8.98	3700
Iron, cast (2.7-3.6% C)	0.89-1.45	6.96-7.35	4000
Steel (carbon and low alloy)	1.93-2.20	7.73-7.87	5100
Stainless steel (18% Cr, 8% Ni)	1.93-2.06	7.65-7.93	5100
Titanium (pure and alloy)	1.06-1.14	4.52	4900
Glass	0.49-0.79	2.38-3.88	4500
Methyl methacrylate	0.024-0.034	1.16	1600
Polyethylene	$1.38\text{-}3.8 \times 10^{-3}$	0.915	530
Rubber	$0.79\text{-}4.1 \times 10^{-5}$	0.99-1.245	46

* See S.H Crandall, and N.C. Dahl, *An Introduction to the Mechanics of Solids*, McGraw Hill, New York, 1959, for a list of references for these constants and a list of these constants in English units.
† Computed from average values of E and ρ.

II. Example Case Study - Change in Rod Length

$x_1 = 0$

$l = 0.1$m

Cross-sectional area A=1cm^2

$AT_{11}(l)$

f=100 lb=445 N

x_1

Metal rod fixed at $x_1 = 0$ and subject to the force f at $x_1 = l$.

6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn

Lecture 16
Page 2 of 17

Steady State: $\dfrac{\partial \delta}{\partial t} = 0$, $F = 0$

$$\dfrac{\partial^2 \delta}{\partial x_1^2} = 0 \Rightarrow \delta = ax_1 + b$$

Boundary Condition at $x_1 = l$: $A\,T_{11}\left(x_1 = l\right) = A\,E\left.\dfrac{\partial \delta}{\partial x_1}\right|_{x_1 = l} = AEa = f$

$$\delta\left(x_1 = 0\right) = 0 = b$$

$$\delta\left(x_1\right) = \dfrac{f\,x}{A\,E}$$

$$\delta\left(x_1 = l\right) = \dfrac{f\,l}{A\,E}$$

$E = 0.7 \times 10^{11}\,\text{N}/\text{m}^2$ (Aluminum)

$A = 0.0001\,\text{m}^2$, $f = 445\,\text{N}$, $l = 0.1\,\text{m}$

$$\delta\left(x_1 = l\right) \;=\; \dfrac{445\left(.1\right)}{10^{-4}\left(.7 \times 10^{11}\right)} \;=\; 6.36 \times 10^{-6}\,\text{m} \;=\; 6.36\ \mu\text{m}$$

III. Elastic Waves : $F = 0$

A. Wave Equation $\quad \dfrac{\partial^2 \delta}{\partial t^2} = \dfrac{E}{\rho}\,\dfrac{\partial^2 \delta}{\partial x^2}$, $v_p^{\,2} = \dfrac{E}{\rho} \Rightarrow v_p = \sqrt{E/\rho}$

$$\dfrac{\partial^2 \delta}{\partial t^2} = v_p^{\,2}\,\dfrac{\partial^2 \delta}{\partial x^2}$$

$$\delta = \delta_{+}\left(x - v_p t\right) + \delta_{-}\left(x + v_p t\right)$$

6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn

Lecture 16
Page 3 of 17

Superposition: $\delta = \delta_+ \left(x - v_p t \right) = \delta_+ \left(\alpha \right)$, $\alpha = x - v_p t$

$$\frac{\partial \delta}{\partial t} = \frac{\partial \delta}{\partial \alpha} \underbrace{\frac{\partial \alpha}{\partial t}}_{-v_p} = -v_p \frac{\partial \delta}{\partial \alpha}$$

$$\frac{\partial^2 \delta}{\partial t^2} = \frac{\partial}{\partial \alpha} \left(-v_p \frac{\partial \delta}{\partial \alpha} \right) \underbrace{\frac{\partial \alpha}{\partial t}}_{-v_p} = v_p^2 \frac{\partial^2 \delta}{\partial \alpha^2}$$

$$\frac{\partial \delta}{\partial x} = \frac{\partial \delta}{\partial \alpha} \underbrace{\frac{\partial \alpha}{\partial x}}_{1} = \frac{\partial \delta}{\partial \alpha}$$

$$\frac{\partial^2 \delta}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial \delta}{\partial \alpha} \right) = \frac{\partial}{\partial \alpha} \left(\frac{\partial \delta}{\partial \alpha} \right) \underbrace{\frac{\partial \alpha}{\partial x}}_{1} = \frac{\partial^2 \delta}{\partial \alpha^2}$$

$$\frac{\partial^2 \delta}{\partial t^2} = v_p^2 \frac{\partial^2 \delta}{\partial \alpha^2} = v_p^2 \frac{\partial^2 \delta}{\partial x^2} = v_p^2 \frac{\partial^2 \delta}{\partial \alpha^2} \qquad \text{Q.E.D}$$

For $\delta = \delta_- \left(x + v_p t \right)$ let $v_p \rightarrow -v_p$

B. Velocity and Stress Variables

$$v = \frac{\partial \delta}{\partial t}, \quad T = E \frac{\partial \delta}{\partial x}$$

$$\rho \frac{\partial^2 \delta}{\partial t^2} = \rho \frac{\partial v}{\partial t} = \frac{\partial T}{\partial x}$$

$$\frac{\partial T}{\partial t} = E \frac{\partial^2 \delta}{\partial x \partial t} = E \frac{\partial v}{\partial x}$$

$$\frac{\partial}{\partial t} \bigg| \quad \rho \frac{\partial v}{\partial t} = \frac{\partial T}{\partial x}$$

$$\Rightarrow \frac{\partial^2 T}{\partial x \partial t} = \rho \frac{\partial^2 v}{\partial t^2} = E \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial}{\partial x} \bigg| \quad \frac{\partial T}{\partial t} = E \frac{\partial v}{\partial x}$$

$$\frac{\partial^2 v}{\partial t^2} = v_p^2 \frac{\partial^2 v}{\partial x^2}, \quad v_p^2 = E/\rho$$

6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn

Lecture 16
Page 4 of 17

$$\frac{\partial}{\partial x}\bigg|\ \frac{\partial v}{\partial t} = \frac{1}{\rho}\frac{\partial T}{\partial x}$$

$$\Rightarrow \frac{1}{\rho}\ \frac{\partial^2 T}{\partial x^2} = \frac{1}{E}\ \frac{\partial^2 T}{\partial t^2}$$

$$\frac{\partial}{\partial t}\bigg|\ \frac{\partial v}{\partial x} = \frac{1}{E}\ \frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial t^2} = v_p^2\ \frac{\partial^2 T}{\partial x^2},\ v_p^2 = E/\rho$$

$$v = v_+\left(x - v_p t\right) + v_-\left(x + v_p t\right)$$

$$T = T_+\left(x - v_p t\right) + T_-\left(x + v_p t\right)$$

C. Method of Characteristics

$$\lambda_1\bigg|\ \rho\ \frac{\partial v}{\partial t}\ +\ 0\frac{\partial v}{\partial x}\ -\ 0\ \frac{\partial T}{\partial t}\ -\ \frac{\partial T}{\partial x} = 0$$

Lagrange Multipliers

$$\lambda_2\bigg|\ 0\ \frac{\partial v}{\partial t}\ -\ E\frac{\partial v}{\partial x}\ +\ \frac{\partial T}{\partial t}\ +\ 0\ \frac{\partial T}{\partial t} = 0$$

$$\rho\lambda_1 \underbrace{\frac{\partial v}{\partial t} - \frac{\lambda_2 E}{\rho\lambda_1}\frac{\partial v}{\partial x}}_{\frac{dv}{dt}}\ +\ \underbrace{\frac{\lambda_2}{\rho\lambda_1}\frac{\partial T}{\partial t} - \frac{\lambda_1}{\rho\lambda_1}\frac{\partial T}{\partial x}}_{\frac{\lambda_2}{\rho\lambda_1}\left(\frac{dT}{dt}\right)} = 0$$

$$dv\left(x,t\right) = \frac{\partial v}{\partial t}\ dt + \frac{\partial v}{\partial x}\ dx$$

$$\Rightarrow \frac{dx}{dt} = \frac{-\lambda_2 E}{\rho\lambda_1} = \frac{-\lambda_1}{\lambda_2}$$

$$dT\left(x,t\right) = \frac{\partial T}{\partial t}\ dt + \frac{\partial T}{\partial x}\ dx$$

$$\left(\frac{\lambda_2}{\lambda_1}\right)^2 = \frac{\rho}{E} \Rightarrow \frac{\lambda_2}{\lambda_1} = \pm\sqrt{\frac{\rho}{E}}$$

6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn

Lecture 16
Page 5 of 17

$$\frac{dv}{dt} + \frac{\lambda_2}{\rho\lambda_1}\frac{dT}{dt} = 0 \Rightarrow dv \pm \frac{1}{\sqrt{\rho E}} dT = 0$$

$$dv + \frac{1}{\sqrt{\rho E}} dT = 0 \quad \text{on} \quad \frac{dx}{dt} = -\sqrt{E/\rho} = -v_p$$

$$dv - \frac{1}{\sqrt{\rho E}} dT = 0 \quad \text{on} \quad \frac{dx}{dt} = +\sqrt{E/\rho} = +v_p$$

$$v + \frac{1}{\sqrt{\rho E}} T = C_- \quad \text{on} \quad \frac{dx}{dt} = -v_p$$

$$v - \frac{1}{\sqrt{\rho E}} T = C_+ \quad \text{on} \quad \frac{dx}{dt} = +v_p$$

$$v = \frac{C_+ + C_-}{2}$$

$$\frac{T}{\sqrt{\rho E}} = \frac{C_- - C_+}{2}$$

D.   "Cook-Book" Method

$$dv(x,t) = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial t}dt \quad , \quad \rho\frac{\partial v}{\partial t} = \frac{\partial T}{\partial x}$$

$$dT(x,t) = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial t}dt \quad , \quad \frac{\partial T}{\partial t} = E\frac{\partial v}{\partial x}$$

$$\begin{bmatrix} \rho & 0 & 0 & -1 \\ 0 & -E & 1 & 0 \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{bmatrix} \begin{bmatrix} \partial v/\partial t \\ \partial v/\partial x \\ \partial T/\partial t \\ \partial T/\partial x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ dv \\ dT \end{bmatrix}$$

$$\frac{\partial v}{\partial t} = \frac{\det \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & -E & 1 & 0 \\ dv & dx & 0 & 0 \\ dT & 0 & dt & dx \end{bmatrix}}{\det \begin{bmatrix} \rho & 0 & 0 & -1 \\ 0 & -E & 1 & 0 \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{bmatrix}}$$

set each determinant to zero for general solution

1st Characteristic equation

$$\det \begin{bmatrix} \rho & 0 & 0 & -1 \\ 0 & -E & 1 & 0 \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{bmatrix} = 0$$

$$\rho \begin{bmatrix} -E & 1 & 0 \\ dx & 0 & 0 \\ 0 & dt & dx \end{bmatrix} + dt \begin{bmatrix} 0 & 0 & -1 \\ -E & 1 & 0 \\ 0 & dt & dx \end{bmatrix}$$

$$= \rho \, dx \begin{bmatrix} -E & 1 \\ dx & 0 \end{bmatrix} - dt \begin{bmatrix} -E & 1 \\ 0 & dt \end{bmatrix}$$

$$= -(dx)^2 \rho + E(dt)^2$$

$$\left(\frac{dx}{dt}\right)^2 = \frac{E}{\rho} \Rightarrow \frac{dx}{dt} = \pm\sqrt{E/\rho} = \pm v_p$$

6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn

Lecture 16
Page 7 of 17

2nd Characteristic equation

$$\det \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & -E & 1 & 0 \\ dv & dx & 0 & 0 \\ dT & 0 & dt & dx \end{bmatrix} = \begin{bmatrix} 0 & -E & 1 \\ dv & dx & 0 \\ dT & 0 & dt \end{bmatrix}$$

$$= E \begin{bmatrix} dv & 0 \\ dT & dt \end{bmatrix} + \begin{bmatrix} dv & dx \\ dT & 0 \end{bmatrix}$$

$$= E\, dv\, dt - dx\, dT = 0$$

$$E\, dv - dT\, \frac{dx}{dt} = 0 \quad \Rightarrow \quad E\, dv \mp \sqrt{\frac{E}{\rho}}\, dT = 0$$
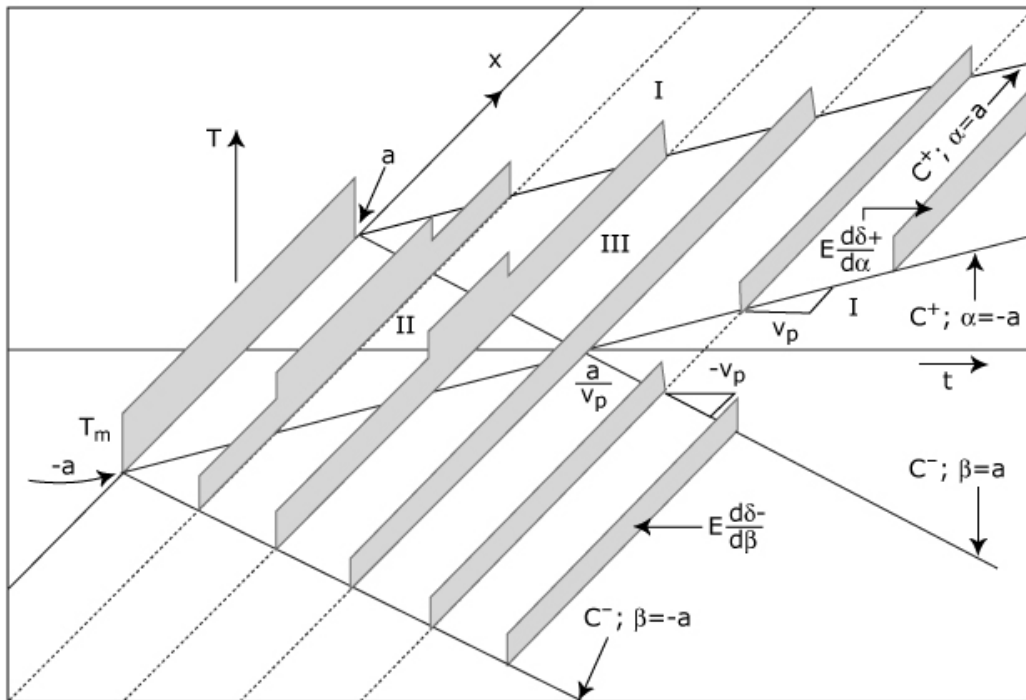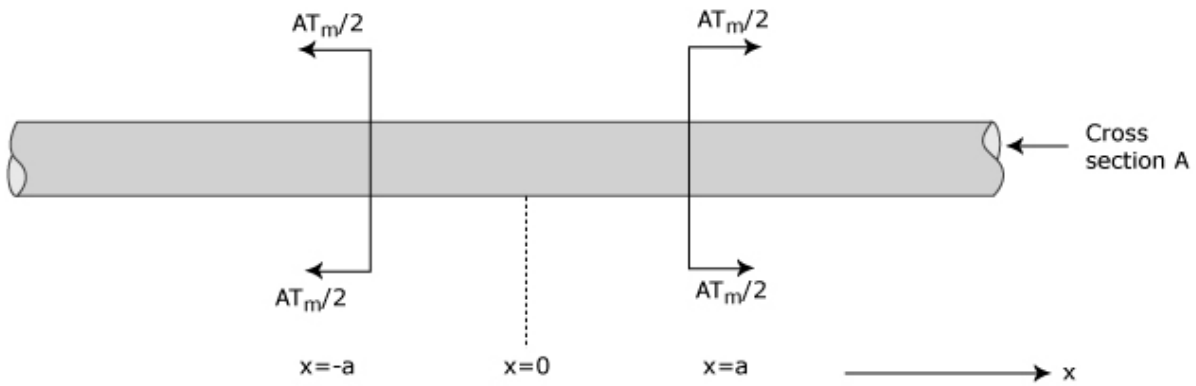
$$dv \mp \frac{dT}{\sqrt{\rho E}} = 0 \quad \Rightarrow \quad v \mp \frac{T}{\sqrt{\rho E}} = \text{constant}$$
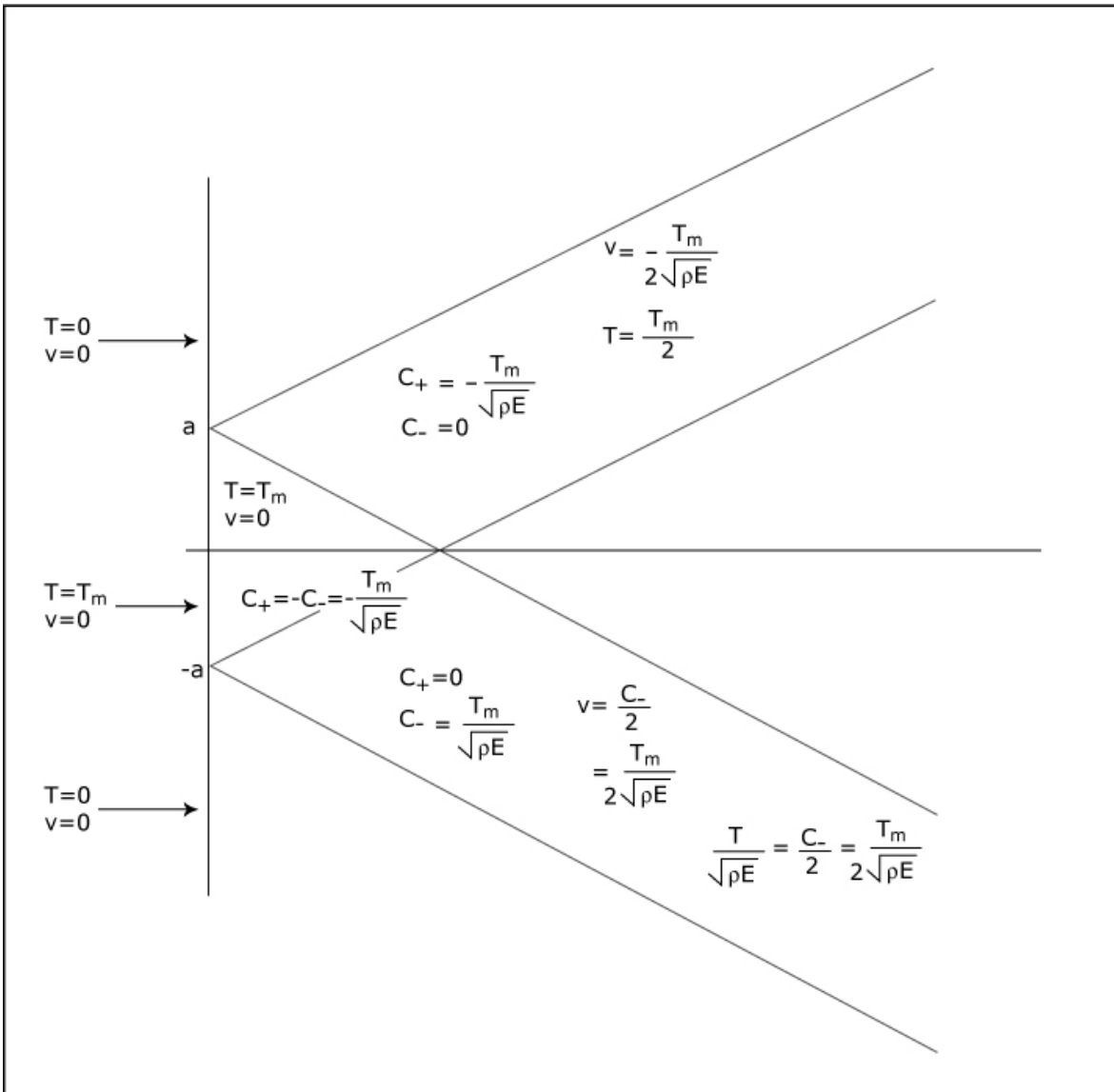
$$v - \frac{T}{\sqrt{\rho E}} = C_+ \quad \text{on} \quad \frac{dx}{dt} = +v_p$$

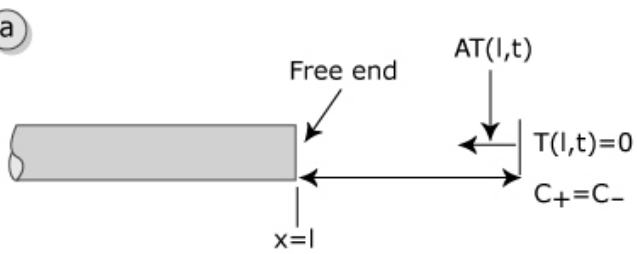$$v + \frac{T}{\sqrt{\rho E}} = C_- \quad \text{on} \quad \frac{dx}{dt} = -v_p$$
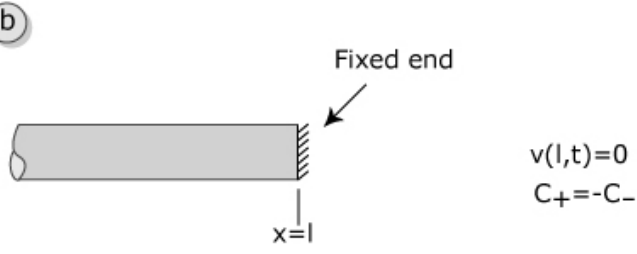
IV. Case Study : Region of Initial Uniform Stress

$$t = 0: \quad T(x, t = 0) = \begin{cases} T_m & |x| < a \\ \\ 0 & |x| > a \end{cases}$$
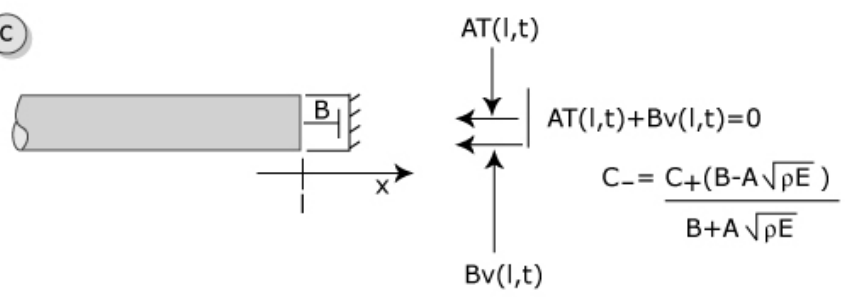
Thin rod subject to an initial uniform static stress T_m over the section $-a<x<a$. At t=0 the external forces $AT_m/2$ are removed.

Stress distribution T as a function of x at succeeding instants in time. When t = 0, the stress is uniform between x = a and x = -a and zero elsewhere; the velocity is zero.

6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn

Lecture 16
Page 9 of 17

$$v = -\frac{T_m}{2\sqrt{\rho E}}$$

$$T = \frac{T_m}{2}$$

T=0
v=0

$$C_+ = -\frac{T_m}{\sqrt{\rho E}}$$

$$C_- = 0$$

a

$$T=T_m$$
$$v=0$$

$$C_+ = -C_- = -\frac{T_m}{\sqrt{\rho E}}$$

$$T=T_m$$
$$v=0$$

-a

$$C_+ = 0$$

$$C_- = \frac{T_m}{\sqrt{\rho E}}$$

$$v = \frac{C_-}{2}$$

$$= \frac{T_m}{2\sqrt{\rho E}}$$

T=0
v=0

$$\frac{T}{\sqrt{\rho E}} = \frac{C_-}{2} = \frac{T_m}{2\sqrt{\rho E}}$$

VIII. Boundary Conditions

Simple boundary conditions on the end of a thin rod: (a) free end; (b) fixed end; (c) end attached to a damper producing a total force Bv.
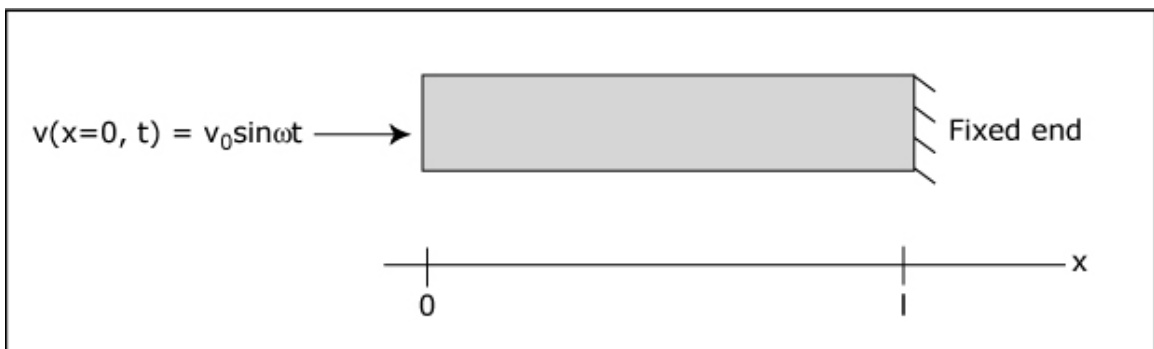
6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn

Lecture 16
Page 11 of 17

(a) Thin rod of length 2l centered at x=0; (b) an x-t plot showing the characteristics relevant to the effect of the ends on the dynamics.

Propagation of an initial pulse of stress on a rod terminated at $x = \pm l$ with free ends.
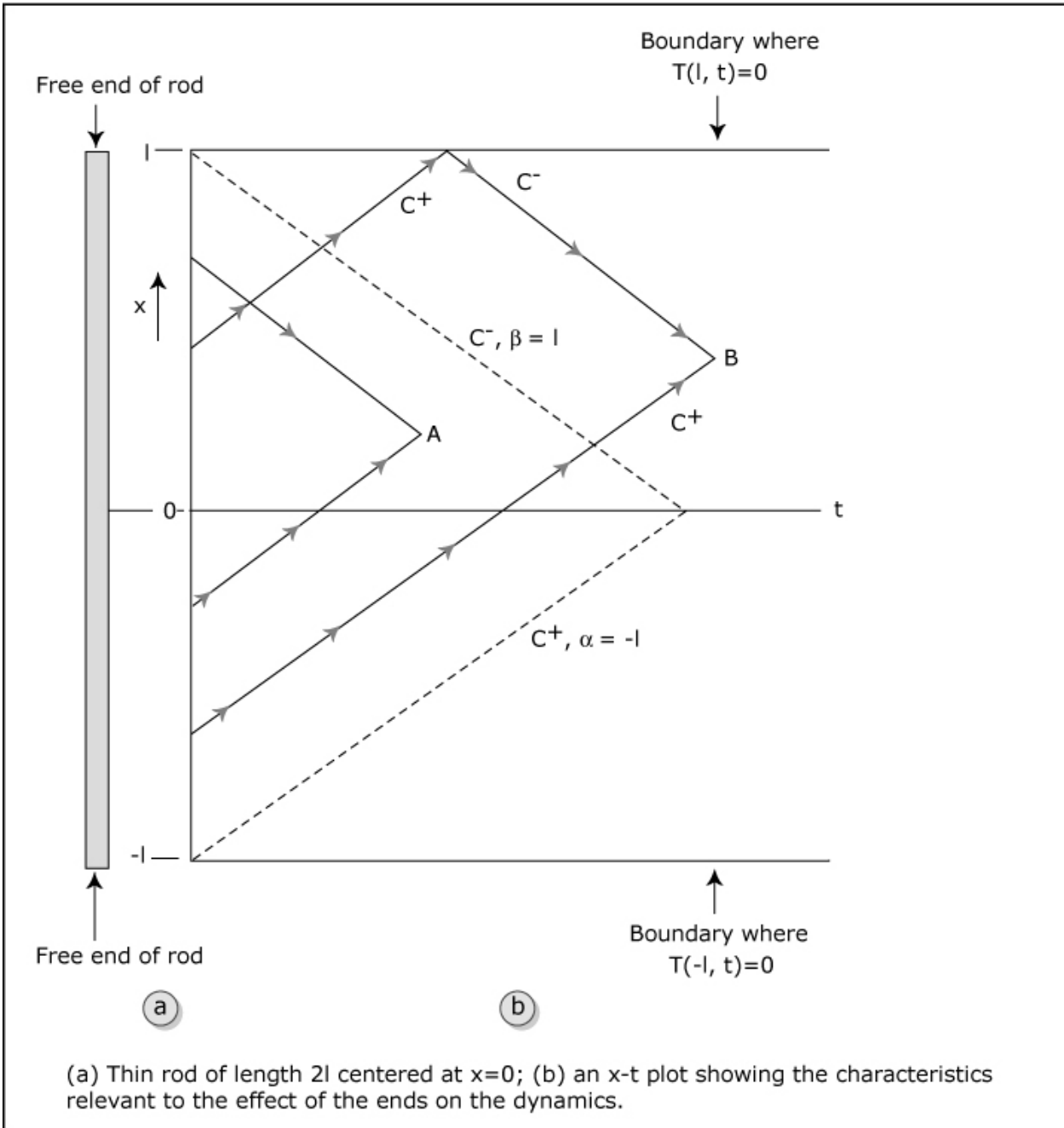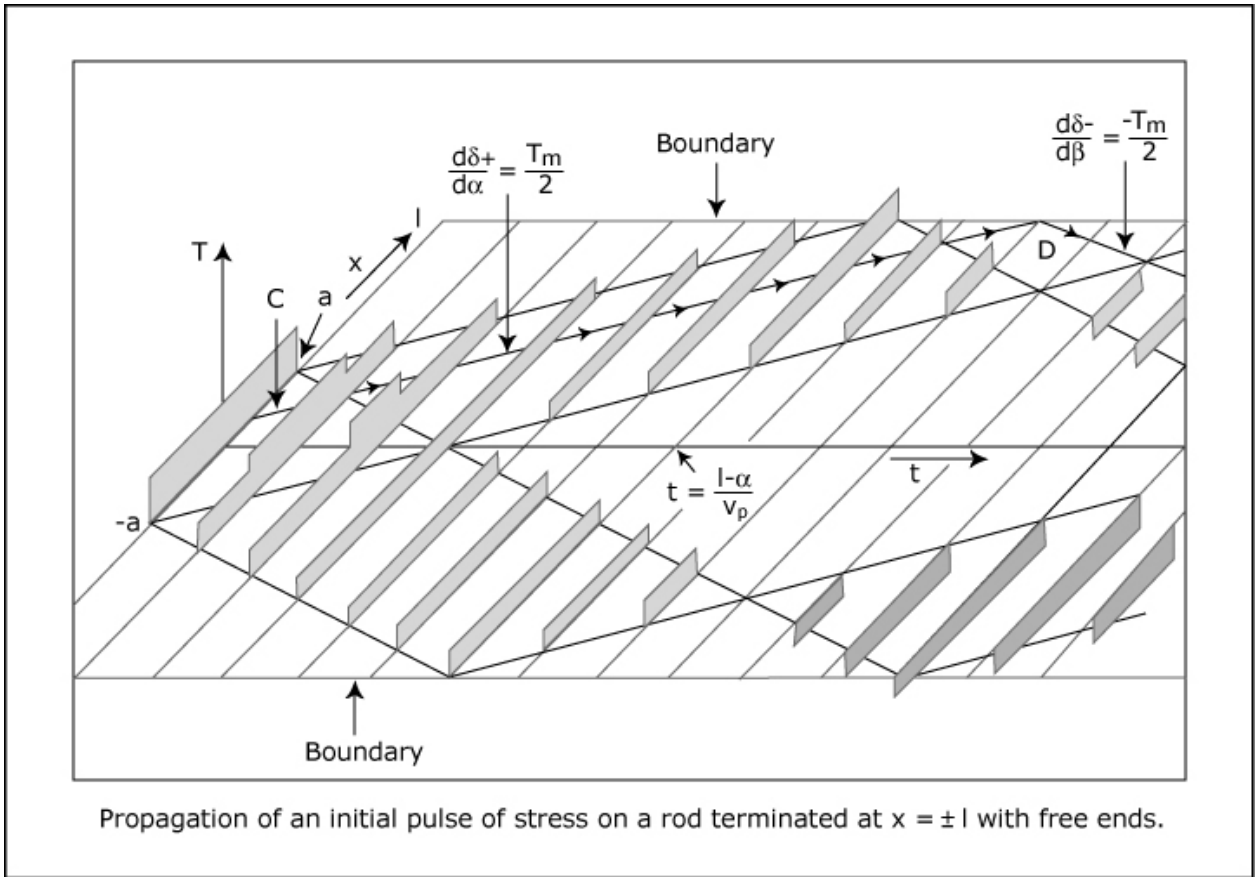
IX. Sinusoidal Steady State

    A.   Velocity Driven Thin Rod

$$v(x,t) = \text{Re}\left[\hat{v}(x)e^{j\omega t}\right]$$

B.C: $\hat{v}(x=0) = -jv_0$

6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn

Lecture 16
Page 13 of 17

$$\hat{v}\left(x = l\right) = 0$$

$$\frac{\partial^2 v}{\partial t^2} = v_p^2 \; \frac{\partial^2 v}{\partial x^2} \quad \Rightarrow \quad -\omega^2 \, \hat{v}(x) = \frac{d^2 \hat{v}}{dx^2} \, v_p^2$$

$$\frac{d^2 \hat{v}}{dx^2} + k^2 \, \hat{v}\left(x\right) = 0, \; k^2 = \omega^2 / v_p^2 = \frac{\rho \, \omega^2}{E}$$

$$\hat{v}\left(x\right) = A \sin kx + B \cos kx$$

$$\hat{v}\left(x = 0\right) = B = -j v_0$$

$$\hat{v}\left(x = l\right) = 0 = A \sin kl - j v_0 \, \cos kl \Rightarrow A = j v_0 \cot kl$$

$$\hat{v}\left(x\right) = j v_0 \left(\frac{\cos kl}{\sin kl} \, \sin kx - \cos kx \right)$$

$$= \frac{j v_0}{\sin kl} \left(\cos kl \, \sin kx - \cos kx \, \sin kl \right)$$

$$= \frac{j v_0 \sin k(x - l)}{\sin kl}$$

$$\frac{\partial T}{\partial t} = E \frac{\partial v}{\partial x} \Rightarrow \cancel{j} \omega \hat{T}\left(x\right) = \cancel{j} v_0 \, E \, k \frac{\cos k\left(x - l\right)}{\sin kl}$$

$$\hat{T}\left(x\right) = v_0 \, E \, \frac{k}{\omega} \, \frac{\cos k\left(x - l\right)}{\sin kl}$$

$$= v_0 \, E \sqrt{\rho / E} \, \frac{\cos k\left(x - l\right)}{\sin kl}$$

$$= v_0 \sqrt{\rho E} \, \frac{\cos k\left(x - l\right)}{\sin kl}$$

$$v\left(x, t\right) = \mathrm{Re}\left[\hat{v}\left(x\right) e^{j\omega t} \right] = -v_0 \, \frac{\sin k\left(x - l\right)}{\sin kl} \, \sin \omega t$$

6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn

Lecture 16
Page 14 of 17

$$T(x,t) = \text{Re}\left[\hat{T}(x)e^{j\omega t}\right] = \frac{V_0\sqrt{\rho E}}{\sin kl}\cos k(x-l)\cos\omega t$$

$$\text{Resonance:} \quad \sin kl = 0 \Rightarrow kl = n\pi \Rightarrow k = \frac{\omega}{v_p} = \frac{n\pi}{l}$$

X. Electromechanical Coupling
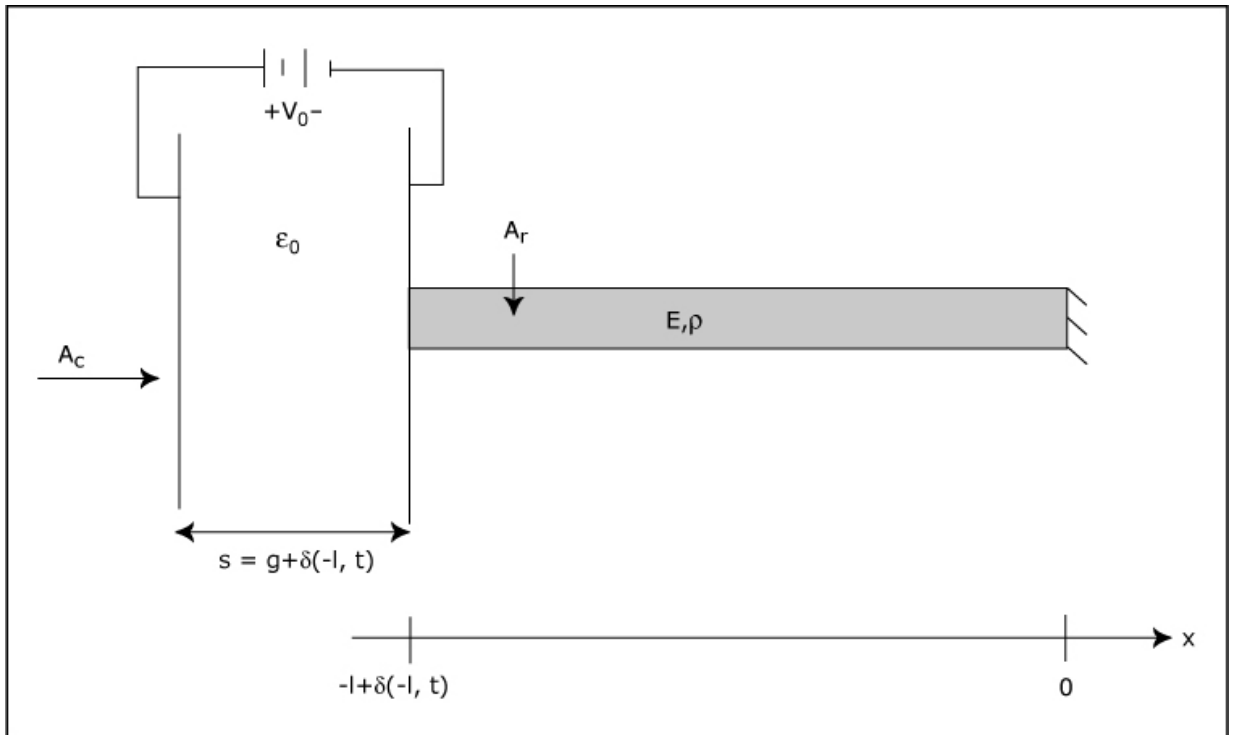
$$C = \frac{\varepsilon_0 A_c}{s} \quad, \quad f_x = \frac{1}{2}v^2\frac{dC}{ds} = -\frac{1}{2}\frac{v^2\varepsilon_0 A_c}{s^2}$$

f_x has steady part & transient part

$$= -\frac{1}{2}\frac{V_0^2\,\varepsilon_0 A_c}{\left(g+\delta(-l,t)\right)^2}$$

$$\delta = \delta_0(x) + \delta'(x,t)$$

$$= \frac{-\dfrac{1}{2}\varepsilon_0 A_c V_0^2}{g^2\left(1+\dfrac{\delta(-l,t)}{g}\right)^2}$$

Steady part $\left(\dfrac{\partial}{\partial t}\delta_0 = 0\right)$

$$\approx \frac{-\dfrac{1}{2}\varepsilon_0 A_c V_0^2}{g^2\left(1+\dfrac{2\delta(-l,t)}{g}\right)}$$

6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn

Le cture 16
Page 15 of 17

$$\approx \frac{\frac{-1}{2} \varepsilon_0 A_c V_0^2}{g^2} \left(1 - \frac{2\delta(-l, t)}{g}\right)$$

$$\frac{\partial^2 \delta_0}{\partial x^2} = 0 \implies \delta_0 = ax + b$$

$$\delta_0 (x = 0) = b = 0$$

$$\delta_0 = ax$$

$$E A_r \left. \frac{\partial \delta_0}{\partial x}\right|_{x=-l} = +\frac{1}{2} \frac{\varepsilon_0 A_c V_0^2}{g^2} \left[1 + \frac{2 a l}{g}\right]$$

$$= E A_r a$$

$$a \left[E A_r - \frac{\varepsilon_0 A_c V_0^2 l}{g^3}\right] = +\frac{1}{2} \frac{\varepsilon_0 A_c V_0^2}{g^2} \implies a = \frac{+\frac{1}{2} \varepsilon_0 A_c V_0^2}{g^2 \left[E A_r - \frac{\varepsilon_0 A_c V_0^2 l}{g^3}\right]}$$

Transient Part: $\delta'(x, t) = \text{Re}\left[\hat{\delta}(x) e^{j\omega t}\right]$

$$\rho \frac{\partial^2 \delta'}{\partial t^2} = \frac{E \partial^2 \delta'}{\partial x^2} \implies \hat{\delta}'(x) = A_1 \sin kx + A_2 \cos kx, k = \omega/v_p$$
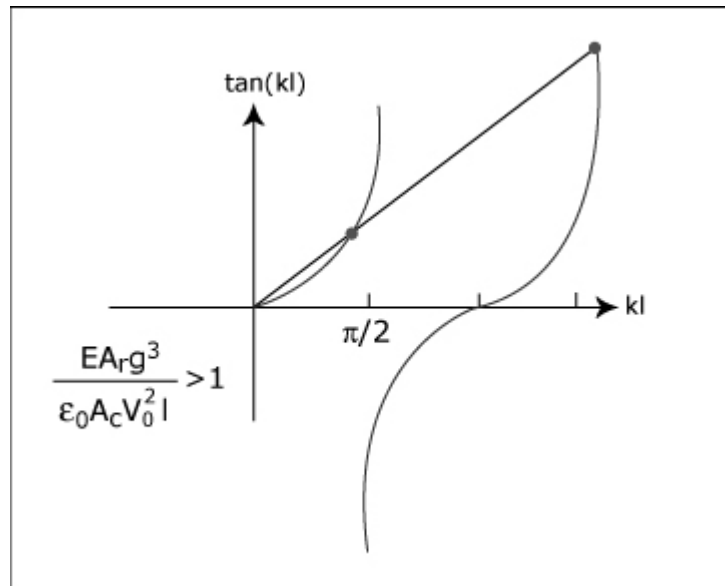
$$\hat{\delta}(x = 0) = 0 = A_2$$

$$\hat{\delta}'(x) = A_1 \sin kx$$

B.C. at $x = -l$

$$E A_r \left. \frac{\partial \delta'}{\partial x}\right|_{x=-l} = -\frac{\varepsilon_0 A_c V_0^2}{g^3} \delta'(-l, t)$$

6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn

Lecture 16
Page 16 of 17

$$E A_r k \cos kl = + \frac{\varepsilon_0 A_c V_0^2}{g^3} \sin kl$$

$$\tan kl = \frac{E A_r g^3}{\varepsilon_0 A_c V_0^2 l} (kl)$$

$$\frac{E A_r g^3}{\varepsilon_0 A_c V_0^2 l} > 1$$

$$\frac{E A_r g^3}{\varepsilon_0 A_c V_0^2 l} < 1 \Rightarrow k = jk_i$$

$$\tan kl = \tan(jk_i l) = j \tanh(k_i l) = jk_i l \left(\frac{E A_r g^3}{\varepsilon_0 A_c V_0^2 l} \right)$$

$$\omega_i = k_i v_p$$

Unstable root

$$e^{j(j\omega_i t)} = e^{-\omega_i t} \quad \text{(Unbounded if } \omega_i < 0\text{)}$$

6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn

Lecture 16
Page 17 of 17