6.642 Continuum Electromechanics
Fall 2008

6.642, Continuum Electromechanics, Fall 2004
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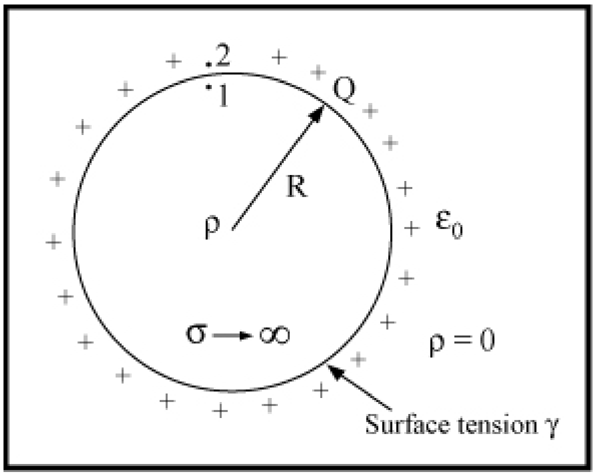
Lecture 10: Stability of a Perfectly Conducting Spherical Drop (Rayleigh's Limit)
Continuum Electromechanics (Melcher) – Section 8.13

I. Geometry-Equilibrium



$$E_r = \frac{Q}{4\pi\varepsilon_0 r^2} = E_0\left(\frac{R}{r}\right)^2 \text{ for } r<R$$

$$E_0 = \frac{Q}{4\pi\varepsilon_0 R^2}$$

$$\bar{T}_s = \frac{-2\gamma}{R}\bar{n}$$

$P_2 = 0$ (air or vacuum, density=0)

$\bar{E}_1 = 0$ (perfectly conducting drop)

$$P_1 - \frac{2\gamma}{R} + \frac{1}{2}\varepsilon_0 E_r\left(r=R\right)^2 = 0$$

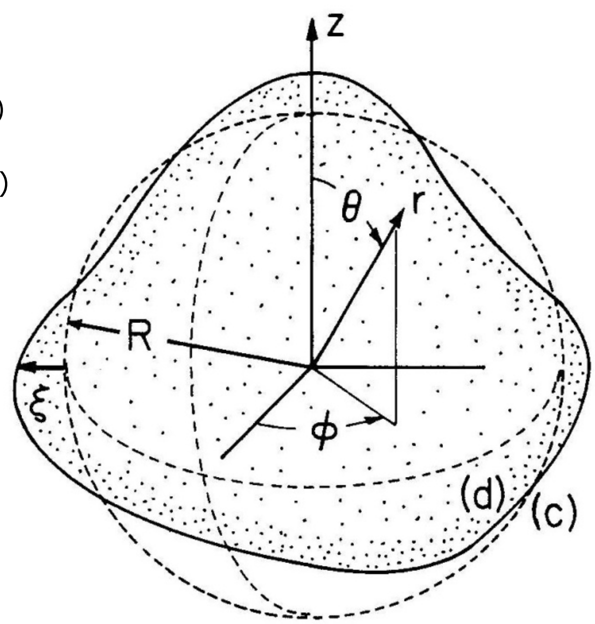
II. Perturbations

$$\xi = \mathrm{Re}\left[\hat{\xi}\, P_n^m\left(\cos\theta\right) e^{j(\omega t - m\phi)}\right]$$

$$F = \xi\left(\theta, \phi, t\right) - r + R$$

$$\bar{n} = \frac{\nabla F}{|\nabla F|} = \bar{i}_r - \frac{1}{R}\frac{\partial\xi}{\partial\theta}\bar{i}_\theta - \frac{1}{R\sin\theta}\frac{\partial\xi}{\partial\phi}\bar{i}_\phi$$

$$\bar{T}_s = -\gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\bar{n} = -\gamma\left(\nabla\cdot\bar{n}\right)\bar{n}$$



Spherically symmetric equilibrium for a drop having total charge q uniformly distributed over its surface.

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$$\hat{T}_{sr} = -\frac{\gamma}{R^2}(n-1)(n+2)\hat{\xi}$$

$\hat{e}_{r1} = 0$ (perfectly conducting drop)

$$\hat{e}_{r2} = \frac{(n+1)\hat{\Phi}_2}{R}$$

$\hat{P}_2 = 0$ (air or vacuum, density=0)

$$\hat{P}_1 = \frac{\rho\omega^2 R}{n}\hat{\xi}$$

III. Electric Field Boundary Conditions

$$\bar{n}\times\bar{E}\Big|_{r=R+\xi} = 0 \;\Rightarrow\; \left[\bar{i}_r - \frac{1}{R}\frac{\partial\xi}{\partial\theta}\bar{i}_\theta - \frac{1}{R\sin\theta}\frac{\partial\xi}{\partial\phi}\bar{i}_\phi\right]\times\left[\left(E_0(R+\xi)+e_r(R)\right)\bar{i}_r + e_\theta(R)\,\bar{i}_\theta + e_\phi(R)\,\bar{i}_\phi\right] = 0$$

$$\bar{i}_\phi\left[e_\theta + \frac{E_0}{R}(R)\frac{\partial\xi}{\partial\theta}\right] + \bar{i}_\theta\left[-e_\phi - \frac{1}{R\sin\theta}E_0(R)\frac{\partial\xi}{\partial\phi}\right] = 0$$

$$\left.\begin{aligned} e_\theta &= -\frac{1}{R}\frac{\partial\Phi_2}{\partial\theta} = -\frac{E_0(R)}{R}\frac{\partial\xi}{\partial\theta} \\ e_\phi &= -\frac{1}{R\sin\theta}\frac{\partial\Phi_2}{\partial\phi} = -\frac{E_0(R)}{R\sin\theta}\frac{\partial\xi}{\partial\phi}\end{aligned}\right\}\Rightarrow \hat{\Phi}_2 = E_0(R)\,\hat{\xi}$$

IV. Interfacial Force Balance Boundary Conditions

$$P_{10} - P_{20} + \hat{P}'_1 - \underbrace{\cancel{\hat{P}'_2}}_{0} + T_{rj}n_j - \frac{2\gamma}{R} - \frac{\gamma}{R^2}(n-1)(n+2)\hat{\xi} = 0$$

$$T_{ij} = \varepsilon_0 E_i E_j - \frac{1}{2}\delta_{ij}\varepsilon_0 E_K E_K$$

$$T_{rj}n_j = T_{rr}n_r + T_{r\theta}n_\theta + T_{r\phi}n_\phi$$

$$T_{rr} = \frac{1}{2}\varepsilon_0\left[E_r^2 \underbrace{-\cancel{e_\theta^2} - \cancel{e_\phi^2}}_{\text{Second order}}\right] = \frac{1}{2}\varepsilon_0\left[\left(E_r(r=R) + \frac{dE_r}{dr}\Big|_{r=R}\xi + e_r\right)^2\right]$$

$$= \frac{1}{2}\varepsilon_0\left[E_0^2 + 2\left(\frac{dE_r}{dr}\Big|_{r=R}\xi + e_r\right)E_0\right]$$

$$\left.\begin{aligned} T_{r\theta} &= \varepsilon_0 E_r e_\theta \;\Rightarrow\; T_{r\theta} n_\theta \\ T_{r\phi} &= \varepsilon_0 E_r e_\phi \;\Rightarrow\; T_{r\phi} n_\phi \end{aligned}\right\} \text{ second order}$$

$$T_{rj} n_j = \frac{1}{2}\varepsilon_0 E_0^2 + \varepsilon_0 \left[\left.\frac{dE_r}{dr}\right|_{r=R} \xi + e_{r2} \right] E_0$$

$$E_r = \frac{E_0 R^2}{r^2} \Rightarrow \left.\frac{dE_r}{dr}\right|_{r=R} = \frac{-2E_0 R^2}{R^3} = \frac{-2E_0}{R}$$

$$\underbrace{P_{10} - \cancelto{0}{P_{20}} + \frac{1}{2}\varepsilon_0 E_0^2 - \frac{2\gamma}{R}}_{=0 \text{ (Equilibrium)}} + \hat{P}_1 - \cancelto{0}{\hat{P}_2} + \varepsilon_0 E_0 \left[\frac{-2E_0}{R}\hat{\xi} + \hat{e}_{r2} \right] - \frac{\gamma}{R^2}(n-1)(n+2)\hat{\xi} = 0$$

V. <u>Dispersion Relation</u>

$$\frac{\rho\omega^2 R}{n}\hat{\xi} + \varepsilon_0 \left[\frac{-2E_0^2}{R} + \frac{(n+1)E_0^2}{R} \right]\hat{\xi} - \frac{\gamma}{R^2}(n-1)(n+2)\hat{\xi} = 0$$

$$\rho\omega^2 R = \frac{n(n-1)}{R}\left[\frac{\gamma}{R}(n+2) - \varepsilon_0 E_0^2 \right]$$

First unstable when n=2 $\Rightarrow$ $\varepsilon_0 E_0^2 = \dfrac{4\gamma}{R} = \varepsilon_0 \left(\dfrac{Q}{4\pi\varepsilon_0 R^2} \right)^2$

$$Q = 8\pi\sqrt{\varepsilon_0 \gamma R^3} \quad \text{[Rayleigh Limit]}$$