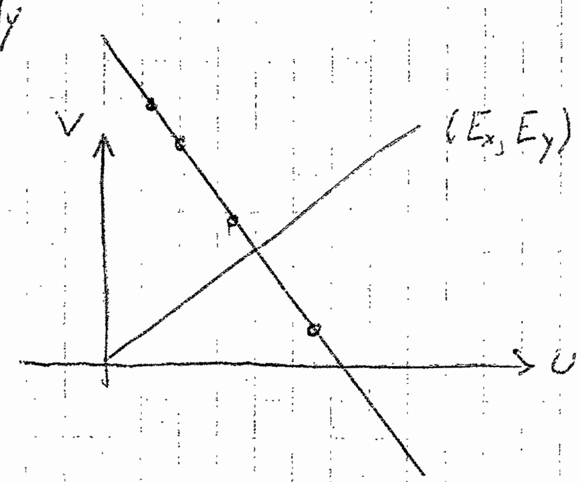
Optical Flow (vs "Fixed flow" u, v constant)

$$uE_x + vE_y + E_t = 0 \qquad u(x,y) \quad v(x,y)$$

let's go! $\min\limits_{u,v} \iint (uE_x + vE_y + E_t)^2 \, dx\,dy$



Euler said: $F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} = 0$

Thus: $\left.\begin{aligned} E_x(uE_x + vE_y + E_t) = 0 \\ E_y(uE_x + vE_y + E_t) = 0 \end{aligned}\right\}$

$\rightarrow$ infinite # of solutions

Ill-posed problem (Hadamard, Tikinov):

(i) no solutions

(ii) infinite # of solutions

(iii) dependent discontinuously on data

What to do $\rightarrow$ add constraints (e.g. smoothness)

Smoothness: the derivative should get smaller as points are closer on the image.

Unsmooth if $u_x^2 + u_y^2 + v_x^2 + v_y^2$ is large. We want to minimize it.

$$\min_{u,v} \iint (uE_x + vE_y + E_t)^2 + \lambda\,(u_x^2 + u_y^2 + v_x^2 + v_y^2)\, dx\,dy$$

for homogeneity!

Picking λ should have low impact on the result.

- $\lambda \to 0$: smoothness constraints disappears $\to \infty$ # sol.
- $\lambda \to \infty$: smoothness is all $\to$ u, v constant

$$\begin{cases} F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} = 0 \\ F_v - \frac{\partial}{\partial x} F_{v_x} - \frac{\partial}{\partial y} F_{v_y} = 0 \end{cases}$$

$$F_u = 2(uE_x + vE_y + E_t)E_x$$
$$F_v = 2(uE_x + vE_y + E_t)E_y$$
$$F_{u_x} = 2\lambda u_x \quad F_{u_y} = 2\lambda u_y$$

Plugging-in ...

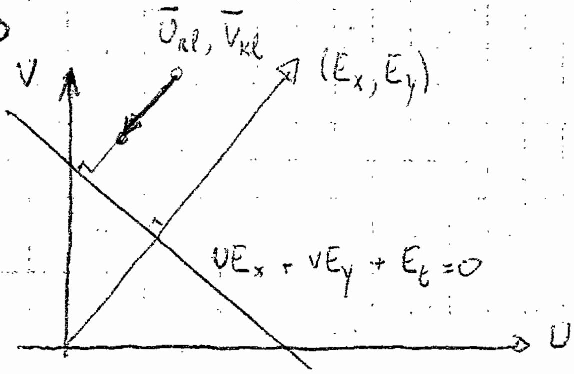
$$\begin{cases} (uE_x + vE_y + E_t)E_x = \nabla^2 u \cdot \lambda \\ (uE_x + vE_y + E_t)E_y = \nabla^2 v \cdot \lambda \end{cases}$$

$$\nabla^2 u = \frac{1}{6\varepsilon^2} \begin{array}{|c|c|c|} \hline 1 & 4 & 1 \\ \hline 4 & -20 & 4 \\ \hline 1 & 4 & 1 \\ \hline \end{array} \qquad \bar{u}_{kl} = \frac{1}{20} \begin{array}{|c|c|c|} \hline 1 & 4 & 1 \\ \hline 4 & & 4 \\ \hline 1 & 4 & 1 \\ \hline \end{array} \qquad \{\nabla^2 u\}_{kl} = \frac{10}{3\varepsilon^2}(\bar{u}_{kl} - u_{kl})$$

Plug-in : $$\begin{cases} -k\lambda(\bar{u}_{kl} - u_{kl}) + (u_{kl}E_x + v_{kl}E_y + E_t)E_x = 0 \\ -k\lambda(\bar{v}_{kl} - v_{kl}) + (u_{kl}E_x + v_{kl}E_y + E_t)E_y = 0 \end{cases}$$

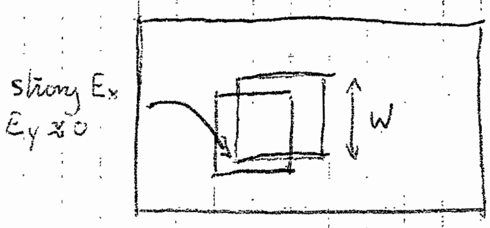
$$\Delta = k\lambda(k\lambda + E_x^2 + E_y^2) \qquad k\lambda > 0$$

...



$$u_{kl} = \bar{u}_{kl} - \frac{\bar{u}_{kl}E_x + \bar{v}_{kl}E_y + E_t}{k\lambda + E_x^2 + E_y^2} \cdot E_x$$

and iterate $u_{kl}^{(n+1)} = f(u_{kl}^{(n)})$



Multi-scale

Sub-sample

ill-posed $\Rightarrow$ "regularization"

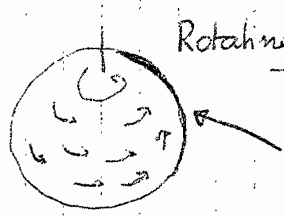
ⓧ OPTICAL FLOW $\longrightarrow$ ill-posed

ⓧ Shape from shading $\longrightarrow$ not ill-posed given singular point

ⓧ Shape from gradient $\longrightarrow$ not-ill-posed (over-determined)

<u>Smoothness Assumption</u> : how valid is that ?

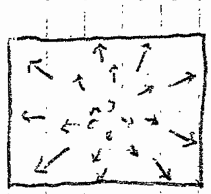
Rotating sphere



discontinuity : flow velocity does not make sense here.

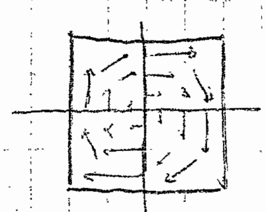
<u>Samples flows</u> ① $u = \alpha x$ $\quad v = \alpha y$ (Approaching a blackboard)

SMOOTH



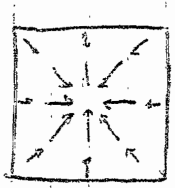
② $u = \alpha y$ $\quad v = -\alpha x$

(Rotation)



SMOOTH !

③ $u = -\frac{1}{\alpha x}$ $\quad v = -\frac{1}{\alpha y}$ (Sink)



no rigid motion

SMOOTH, except near the origin.

④ $u = \frac{1}{\alpha y}$ $\quad v = -\frac{1}{\alpha x}$

(hurricane)