6.852: Distributed Algorithms
Fall, 2009

Class 10

Today's plan

- Simulating synchronous algorithms in asynchronous networks
- Synchronizers
- Lower bound for global synchronization
- Reading: Chapter 16
- Next:
  - Logical time
  - Reading: Chapter 18, [Lamport time, clocks…], [Mattern]

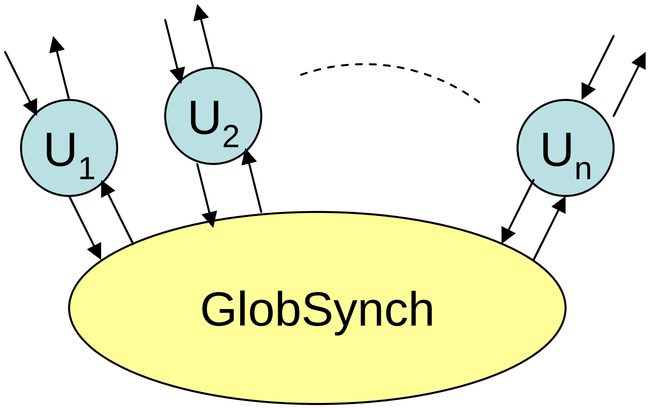
Minimum spanning tree, revisited

- In GHS, complications arise because different parts of the network can be at very different levels at the same time.
- Alternative, more synchronized approach:
  - Keep levels of nearby nodes close, by restricting the asynchrony.
  - Each process uses a level variable to keep track of the level of its current component (according to its local knowledge).
  - Process at level k delays all “interesting” processing until it hears that all its neighbors have reached level ≥ k.
    - Looks (to each process) like global synchronization, but easier to achieve.
    - Each node inform its neighbors whenever it changes level.
- Resulting algorithm is simpler than GHS.
- Complexity:
  - Time: O(n log n), like GHS.
  - Messages: O(|E| log n), somewhat worse than GHS.

Strategy for designing asynchronous distributed algorithms

- Assume undirected graph G = (V,E).
- Design a synchronous algorithm for G, transform it into an asynchronous algorithm using local synchronization.
- Synchronize at every round (not every “level” as above).
- Method works only for non-fault-tolerant algorithms.
  - In fact, no general transformation can work for fault-tolerant algorithms.
  - E.g., ordinary stopping agreement is solvable in synchronous networks, but unsolvable in asynchronous networks [FLP].
- Present a general strategy, some special implementations.
  - Describe in terms of sub-algorithms, modeled as abstract services.
  - [Raynal book], [Awerbuch papers]
- Then a lower bound on the time for global synchronization.
  - Larger than upper bounds for local synchronization.

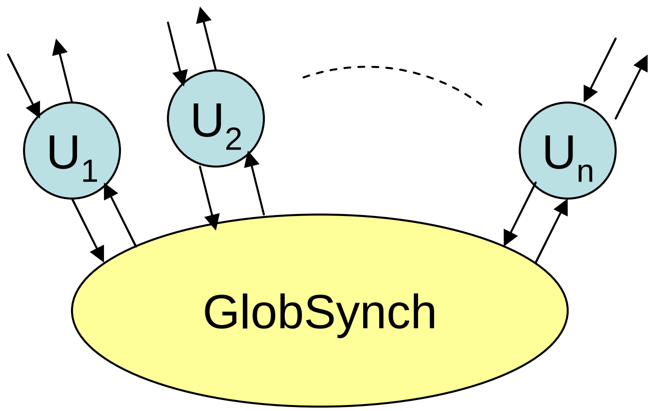
Synchronous model, reformulated in terms of automata



- Global synchronizer automaton
- User process automata:
  - Processes of an algorithm that uses the synchronizer.
  - May have other inputs/outputs, for interacting with other programs.
- Interactions between user process i and synchronizer:
  - user-send$(T,r)_i$
    - T = set of (message, destination) pairs, destinations are neighbors of i.
    - T = empty set $\varnothing$, if no messages sent by i at round r.
    - r = round number
  - user-rcv$(T,r)_i$
    - T = set of (message, source) pairs, source a neighbor of i.
    - r = round number

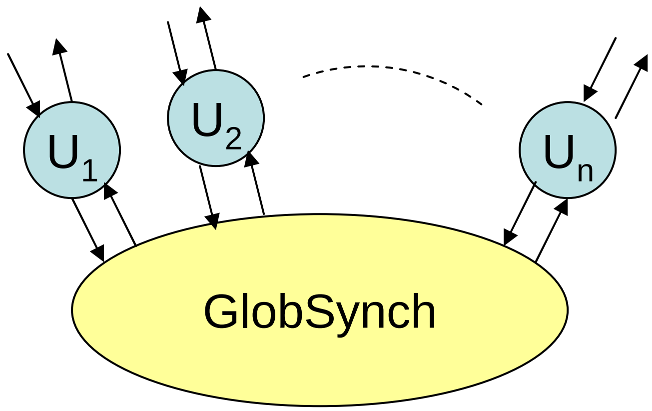
Behavior of GlobSynch

- Manages global synchronization of rounds:
  - Users send packages of all their round 1 messages, using user-send(T,r) actions.
  - GlobSynch waits for all round 1 messages, sorts them, then delivers to users, using user-rcv(T,r) actions.
  - Users send round 2 messages, etc.
- Not exactly the synchronous model:
  - GlobSynch can receive round 2 messages from i before it finishes delivering all the round 1 messages.
  - But it doesn't do anything with these until it's finished round 1 deliveries.
  - So, essentially the same.
- GlobSynch synchronizes globally between each pair of rounds.



Requirements on each U_i

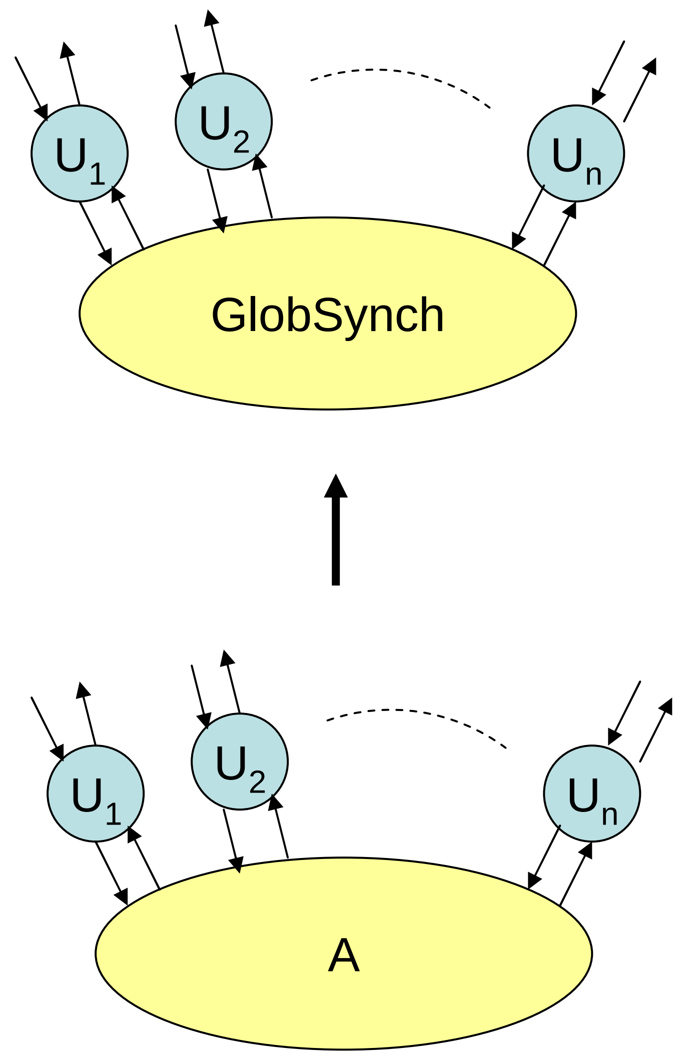
- Well-formed:
  - U_i sends the right kinds of messages, in the right order, at the right times.
- Liveness:
  - After receiving the messages for any round r, U_i eventually submits messages for round r+1.
- See code for GlobSynch in [book, p. 534].
  - State consists of:
    - A tray of messages for each (destination, round).
    - Some Boolean flags to keep track of which sends and rcvs have happened.
  - Transitions obvious.
  - Liveness expressed by tasks, one for each (destination, round).



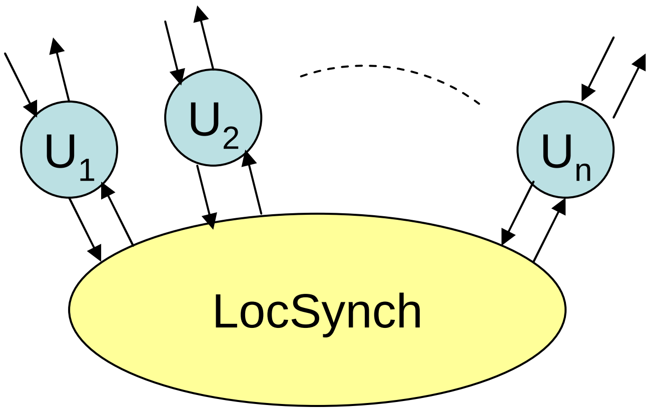
Synchronizers

The Synchronizer Problem

- Design an automaton A that "implements" GlobSynch in the sense that it "looks the same" to each U_i:
  - Has the right interface.
  - Exhibits the right behavior:
    - $\forall$ fair execution α of the U_is and A,
    - $\exists$ fair execution α' of the U_is and GlobSynch, such that
    - $\forall$ i, α is indistinguishable by U_i from α', $\alpha \sim_{U_i} \alpha'$.
- A "behaves like" GlobSynch, as far as any individual U_i can tell.
- Allows global reordering of events at different U_i.



Local Synchronizer, LocSynch



- Enforces local synchronization rather than global, still looks the same locally.
- Only one difference from GlobSynch:
  - Precondition for usr-rcv(T,r)$_i$.
  - Now, to deliver round r messages to user i, check only that i's neighbors have sent round r messages.
  - Don't wait for all nodes to get this far.
- Lemma 1: For every fair execution α of the U_is and LocSynch, there is a fair execution α' of the U_is and GlobSynch, such that for each U_i, $\alpha \sim U_i \; \alpha'$.
- Proof:
  - Can't use a simulation relation, since global order of external events need not be the same, and simulation relations preserve external order.
  - So consider partial order of events and dependencies:

Proof sketch for Lemma 1

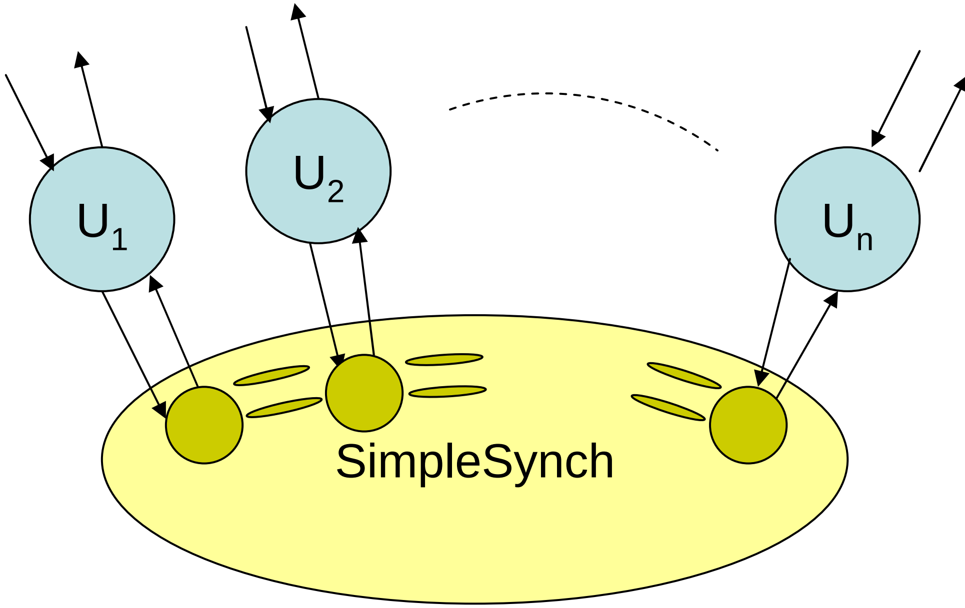
- Consider partial order of events and dependencies:
  - Each U_i event depends on previous U_i events.
  - user-rcv(*,r)$_i$ event depends on user-send(*,r)$_j$ for every neighbor j of i.
  - Take transitive closure.

- Claim: If you start with a (fair) execution of LocSynch system and reorder the events while preserving these dependencies, the result is still a (fair) execution of the LocSynch system.

- So, obtain α' by reordering the events of α so that:
  - These dependencies are preserved, and
  - Events associated with any round r precede those of round r+1.
- Can do this because round r+1 events never depend on round r events.
- This reordering preserves the view of each U_i.
- Also, yields the extra user-rcv precondition needed by GlobSynch.

Trivial distributed algorithm to implement LocSynch

- Processes, point-to-point channels.
- SimpleSynch algorithm, process i:
  - After user-send(T,r)$_i$, send message to each neighbor j containing round number r and any basic algorithm messages i has for j.
  - Send (∅,r) message if i has no basic algorithm messages for j.
  - Wait to receive round r messages from all neighbors.
  - Output user-rcv().
- Lemma 2:
  - For every fair execution α of U_is and SimpleSynch, there is a fair execution α' of U_is and LocSynch, such that for each U_i, $\alpha \sim_{U_i} \alpha'$.
- Here, indistinguishable by all the U_is together---preserves external order.



SimpleSynch, cont'd

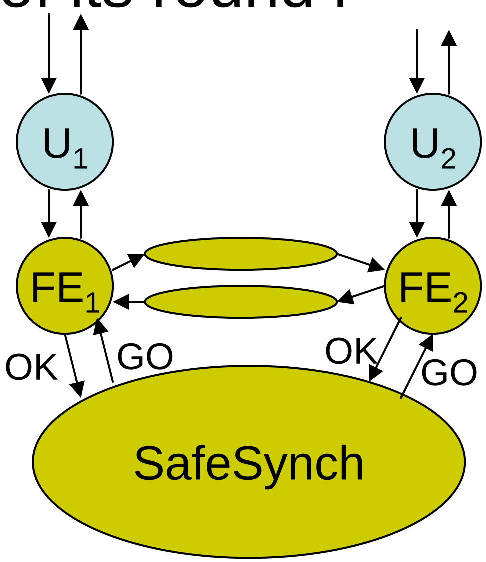
- Proof of Lemma 2:
  - No reordering needed, preserves order of external events.
  - Could use simulation relation.
- Corollary: For every fair execution α of U_is and SimpleSynch, there is a fair execution α' of U_is and GlobSynch, such that for each U_i, $\alpha \sim_{U_i} \alpha'$.
- Proof: Combine Lemmas 1 and 2.
- Complexity:
  - Messages: $\leq 2\ |E|$ per simulated round.
  - Time:
    - Assume user always sends ASAP.
    - l, upper bound on task time for each task of each process.
    - d, upper bound on time for first message in channel to be delivered
    - Then r rounds completed within time r (d + O(l)).

Reducing the communication

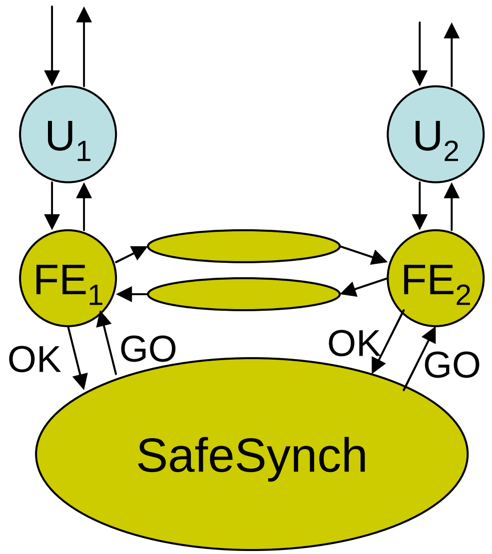
- General Safe Synchronizer strategy [Awerbuch].
- If there's no message $U_i \rightarrow U_j$ at round r of underlying synchronous algorithm, try to avoid sending such messages in the simulating asynchronous algorithm.
- Can't just omit them, since each process must determine, for each round r, when it has received all of its round r messages.
- Approach: Separate the functions of:
  - Sending the actual messages, and
  - Determining when the round is over.
  - Algorithm decomposes into:
    - Front Ends + channels + SafeSynch

For the actual messages

For deciding when finished



Safe Synchronizers



- FE:
  - Sends, receives actual messages for each round r.
  - Sends acks for received messages.
  - Waits to receive acks for its own messages.
- Notes:
  - Sends messages only for actual messages of the underlying algorithm, no dummies.
  - Acks double the messages, but can still be a win.
- FE, cont'd:
  - When FE receives acks for all its round r messages, it's safe: it knows that all its messages have been received by its neighbors.
  - Then sends OK for round r to SafeSynch.
  - Before responding to user, must know that it has received all its neighbors' messages for round r.
  - Suffices to know that all its neighbors are safe, that is, that they know that their messages have been received.
- SafeSynch:
  - Tells each FE when its neighbors are safe!
  - After it has received OK from i and all its neighbors, sends GO to i.

Correctness of SafeSynch

- Lemma 3: For every fair execution α of SafeSynch system, there is a fair execution α' of LocSynch system, such that for each U_i, $\alpha \sim_{U_i} \alpha'$.
- (Actually, indistinguishable to all the U_is together.)

- Corollary: For every fair execution α of SafeSynch system, there is a fair execution α' of GlobSynch system, such that for each U_i, $\alpha \sim_{U_i} \alpha'$.

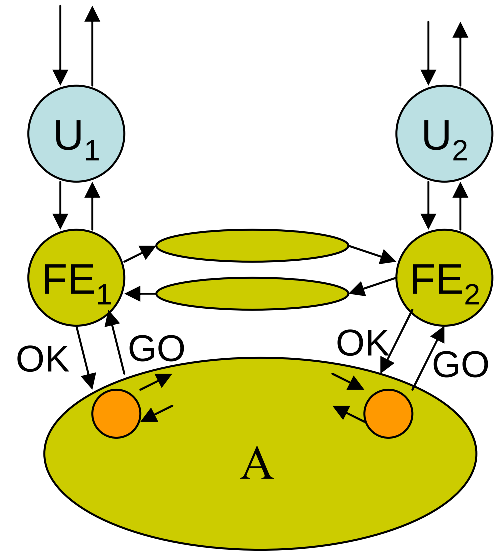
- Must still implement SafeSynch with a distributed algorithm…

- We now give three SafeSynch implementations, Synchronizers A, B, and Γ [Awerbuch].

- All implement SafeSynch, in the sense that the resulting systems are indistinguishable to each U_i (in fact, to all the U_is together).

SafeSynch Implementations

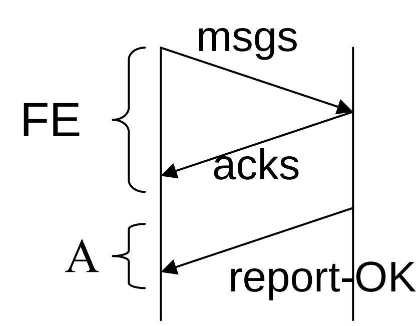
- SafeSynch's job: After receiving OK for round r from i and all its neighbors, send GO for round r to i.



- Synchronizer A:
  - When process i receives OK_i, sends to neighbors.
  - When process i hears that it and all its neighbors have received OKs, outputs GO_i.
- Obviously implements SafeSynch.
- Complexity: To emulate r rounds:
  - Messages: $\le \underbrace{2m}_{\text{Messages and acks by FEs}} + \underbrace{2\, r\, |E|}_{\text{Messages within A}}$, if synch alg sends m actual messages in r rounds.

Messages and acks by FEs

Messages within A



  - Time: $\le r\,(3d + O(l))$

Comparisons

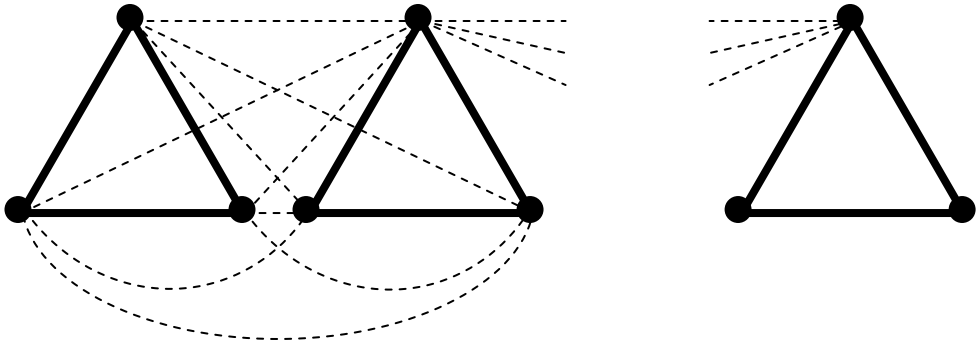
- To emulate r rounds:
  - SafeSynch system with Synchronizer $\mathcal{A}$
    - Messages: 2m + 2 r |E|
    - Time: r (3d + O(l))
  - Simple Synch
    - Messages: 2 r |E|
    - Time: r (d + O(l))
- So Synchronizer $\mathcal{A}$ hasn't improved anything.
- Next, Synchronizer $\mathcal{B}$, with lower message complexity, higher time complexity.
- Then Synchronizer Γ, does well in terms of both messages and time, in an important subclass of networks (those with a "cluster" structure).

Synchronizer $\mathcal{B}$

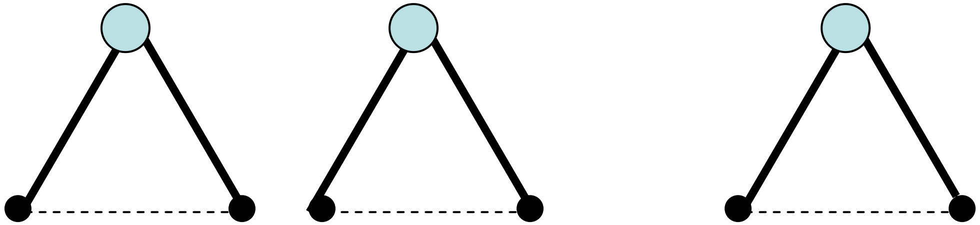
- Assumes rooted spanning tree of graph, height h.
- Algorithm:
  - All processes convergecast OK to root, using spanning tree edges.
  - Root then bcasts permission to GO, again using the spanning tree.
- Obviously implements SafeSynch (overkill).
- Complexity: To emulate r rounds, in which synch algorithm sends m messages:
  - Messages: $\underbrace{2\,m}_{\text{Messages and acks by FEs}} + \underbrace{2\,r\,n}_{\text{Messages within } \mathcal{B}}$
  - Beats $\mathcal{A}$: $2m + 2\,r\,|E|$
  - Time: $\leq r\,(\underbrace{2d + O(l)}_{\text{FEs}} + \underbrace{2h\,(d + O(l))}_{\mathcal{B}\text{, convergecast and broadcast}})$

Synchronizer Γ

- Hybrid of A and B.
- In "clustered" (almost partitionable) graphs, can get performance advantages of both:
  - Time like A, communication like B.
- Assume spanning forest of rooted trees, each tree spanning a "cluster" of nodes.
- Example:
  - Clusters = triangles
  - All edges between adjacent triangles.

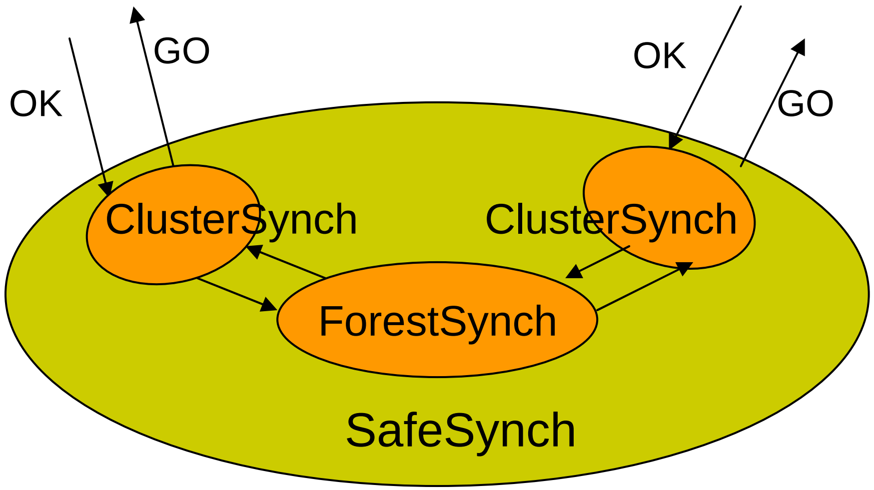


  - Spanning forest:



- Use B within each cluster, A among clusters.

Decomposition of Γ



- ClusterSynch:
  - After receiving OKs from everyone in the cluster, sends cluster-OK to ForestSynch.
  - After receiving cluster-GO from ForestSynch, sends GO to everyone in the cluster.
  - Similar to B.
- ForestSynch:
  - Essentially, a safe synchronizer for the “Cluster Graph” G′:
    - Nodes of G′ are the clusters.
    - Edge between two clusters iff they contain nodes that are adjacent in G.

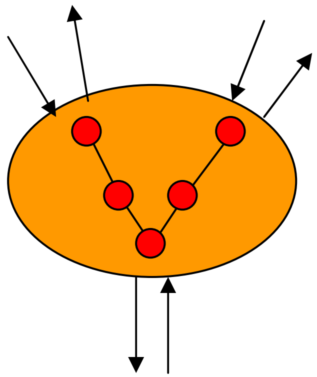
- Lemma: Γ Implements SafeSynch
- Proof idea:
  - Must show: If $GO(r)_i$ occurs, then there must be a previous $OK(r)_{i,}$ and also previous $OK(r)_j$ for every neighbor j of i.

Γ Implements SafeSynch

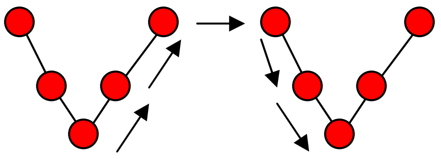
- Show: If $GO(r)_i$ occurs, then there must be a previous $OK(r)_i$, and also previous $OK(r)_j$ for every neighbor j of i.
- Must be a previous $OK(r)_i$:
  - $GO(r)_i$ preceded by cluster-GO(r) for i's cluster (ClusterSynch),
  - Which is preceded by cluster-OK(r) for i's cluster (ForestSynch),
  - Which is preceded by $OK(r)_i$ (ClusterSynch).
- Must be previous $OK(r)_j$ for neighbor j in the same cluster as i.
  - $GO(r)_i$ preceded by cluster-GO(r) for i's cluster (ClusterSynch),
  - Which is preceded by cluster-OK(r) for i's cluster (ForestSynch),
  - Which is preceded by $OK(r)_j$ (ClusterSynch).
- Must be previous $OK(r)_j$ for neighbor j in a different cluster.
  - Then the two clusters are neighboring clusters in the cluster graph G′, because i and j are neighbors in G.
  - $GO(r)_i$ preceded by cluster-GO(r) for i's cluster (ClusterSynch),
  - Which is preceded by cluster-OK(r) for j's cluster (ForestSynch),
  - Which is preceded by $OK(r)_j$ (ClusterSynch).

Implementing ClusterSynch and ForestSynch

- Still need distributed algorithms for these…
- ClusterSynch:
  - Use variant of Synchronizer B on cluster tree:
    - Convergecast OKs to root on the cluster tree,
    - root outputs cluster-OK, receives cluster-GO,
    - root broadcasts GO on the cluster tree.



- ForestSynch:
  - Clusters run Synchronizer A.
    - But clusters can't actually run anything…
    - So cluster roots run A.
    - Simulate communication channels between neighboring clusters by indirect communication paths between the roots.
    - These paths must exist: Run through the trees and across edges that join the clusters.



- cluster-OK and cluster-GO are internal actions of the cluster root processes.

Putting the pieces together

- In Γ, real process i emulates $FrontEnd_i$, process i in ClusterSynch algorithm, and process i in ForestSynch algorithm.
  - Composition of three automata.
- Real channel $C_{i,j}$ emulates channel from $FrontEnd_i$ to $FrontEnd_j$, channel from i to j in ClusterSynch algorithm, and channel from i to j in ForestSynch algorithm.
- Orthogonal decompositions of Γ:
  - Physical: Nodes and channels.
  - Logical: FEs, ClusterSynch, and ForestSynch
  - Same system, 2 views.
  - Works because composition of automata is associative, commutative.
- Such decompositions are common for complex distributed algorithms:
  - Each node runs pieces of algorithms at several layers.

- Theorem 1: For every fair execution α of Γ system (or A, or B), there is a fair execution α' of GlobSynch system, such that for each U_i, $\alpha \sim_{U_i} \alpha'$.

Complexity of Γ

- Consider r rounds, in which the synchronous algorithm sends m messages.
- Let:
  - h = max height of a cluster tree
  - e' = total number of edges on shortest paths between roots of neighboring clusters.
- Messages: $2m + O(r\,(n + e'))$

Messages between roots,
In ForestSynch algorithm

Messages and acks by FEs

Messages in cluster trees,
In ClusterSynch algorithm

- Time: $O(r\,h\,(d + l))$
- If $n + e' \ll |E|$, then Γ's message complexity is much better than A's.
- If h << height of spanning tree of entire network, then Γ's time complexity is much better than B's.
- Both of these are true for “nicely clustered” networks.

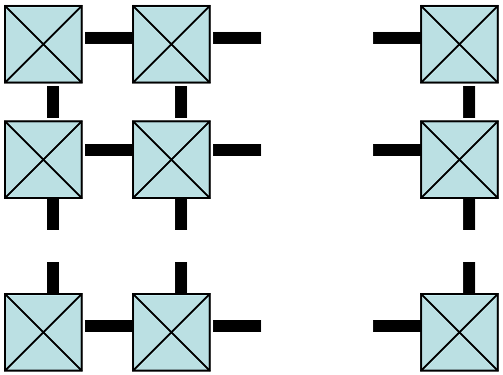
Comparison of Costs

- r rounds
- m messages sent by synchronous algorithm
- d, message delay
- Ignore local processing time l.
- e′ = total length of paths between roots of neighboring clusters
- h = height of global spanning tree
- h′ = max height of cluster tree

	Messages	Time
A	2 m + 2 r |E|	O(r d)
B	2 m + 2 r n	O(r h d)
Γ	2 m + O(r (n + e′))	O(r h′ d)

Example

- $p \times p$ grid of complete k-graphs, with all nodes of neighboring k-graphs connected.
- Clusters = k-graphs
- $h = O(p)$
- $h' = O(1)$



	Messages	Time
A	$2m + O(r\, p^2 k^2)$	$O(r\, d)$
B	$2m + O(r\, p^2 k)$	$O(r\, p\, d)$
Γ	$2m + O(r\, p^2 k)$	$O(r\, d)$

Application 1: Breadth-first search

- Recap:
  - SynchBFS:
    - Constructs BFS tree
    - O(|E|) messages, O(diam) rounds
  - When run in asynchronous network:
    - Constructs a spanning tree, but not necessarily BFS
  - Modified version, with corrections:
    - Constructs BFS tree
    - O(n |E|) messages, O(diam n d) time (counting pileups)
- BFS using synchronizer:
  - Runs more like SynchBFS, avoids corrections, pileups
  - With Synchronizer A:
    - O(diam |E|) messages, O(diam d) time
  - With Synchronizer B :
    - Better communication, but costs time.
  - With Synchronizer Γ :
    - Better overall, in clustered graphs.

Application 2: Broadcast and ack

- Use synchronizer to simulate synchronous broadcast-ack combination.
- Assume known leader, but no spanning tree.
- Recap:
  - Synchronous Bcast-ack:
    - Constructs spanning tree while broadcasting
    - O(|E|) messages, O(diam) rounds
  - Asynchronous Bcast-ack:
    - Timing anomaly: Construct non-min-hop paths, on which acks travel.
    - O(|E|) messages, O(n d) time
- Using (e.g.) Synchronizer $\mathcal{A}$:
    - Avoids timing anomaly.
    - Broadcast travels on min-hop paths, so acks follow min-hop paths.
    - O(diam |E|) messages, O(diam d) time

Application 3: Shortest paths

- Assume weights on edges.
- Without termination detection.
- Recap:
  - Synchronous Bellman-Ford:
    - Allows some corrections, due to low-cost high-hop-count paths.
    - O(n |E|) messages, O(n) rounds
  - Asynch Bellman-Ford
    - Many corrections possible (exponential), due to message delays.
    - Message complexity exponential in n.
    - Time complexity exponential in n, counting message pileups.
- Using (e.g.) Synchronizer A:
    - Behaves like Synchronous Bellman-Ford.
    - Avoids corrections due to message delays.
    - Still has corrections due to low-cost high-hop-count paths.
    - O(n |E|) messages, O(n d) time
    - Big improvement.

Further work

- To read more:
  - See Awerbuch’s extensive work on
    - Applications of synchronizers.
    - Distributed algorithms for clustered networks.
  - Also work by Peleg
- Q: This work used a strategy of purposely slowing down portions of a system in order to improve overall performance. In which situations is this strategy a win?

Lower Bound on Time for Synchronization

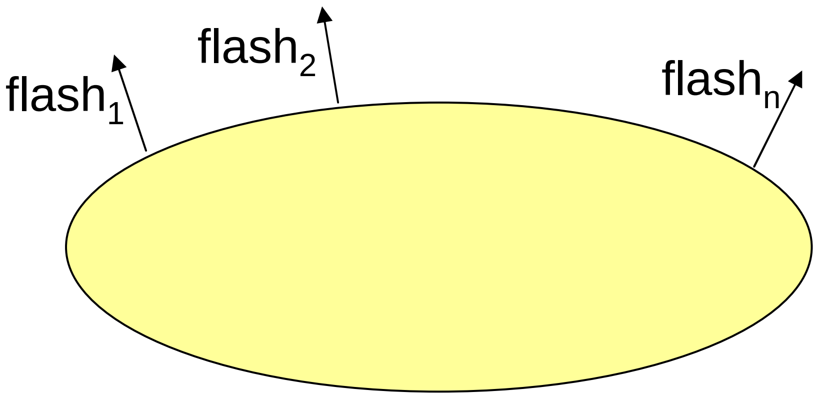
Lower bound on time

- A, B, Γ emulate synchronous algorithms only in a local sense:
  - Looks the same to individual users,
  - Not to the combination of all users---can reorder events at different users.
- Good enough for many applications (e.g., data management).
- Not for others (e.g., embedded systems).

- Now show that global synchronization is inherently more costly than local synchronization, in terms of time complexity.
- Approach:
  - Define a particular global synchronization problem, the k-Session Problem.
  - Show this problem has a fast synchronous algorithm, that is, a fast algorithm using GlobSynch.
    - Time O(k d), assuming GlobSynch takes steps ASAP.
  - Prove that all asynchronous distributed algorithms for this problem are slow.
    - Time Ω(k diam d).
  - Implies GlobSynch has no fast distributed implementation.
- Contrast:
  - A, SimpleSynch are fast distributed implementations of LocSynch.

k-Session Problem

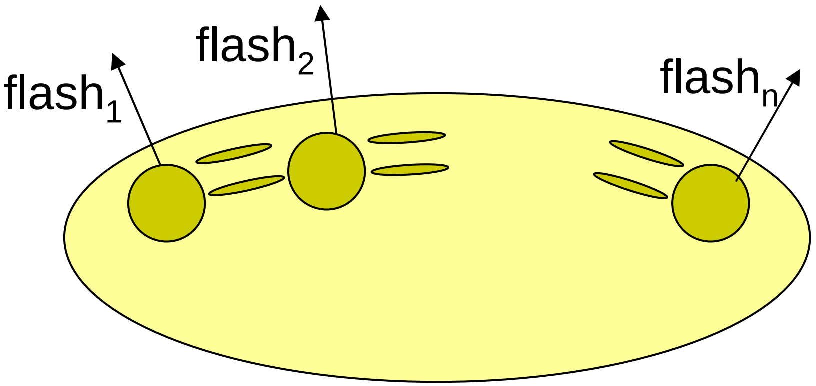
- Session:
  - Any sequence of flash events containing at least one $flash_i$ event for each location i.



- k-Session problem:
  - Perform at least k separate sessions (in every fair execution), and eventually halt.
- Original motivation:
  - Synchronization needed to perform parallel matrix computations that require enough interleaving of process steps, but tolerate extra steps.

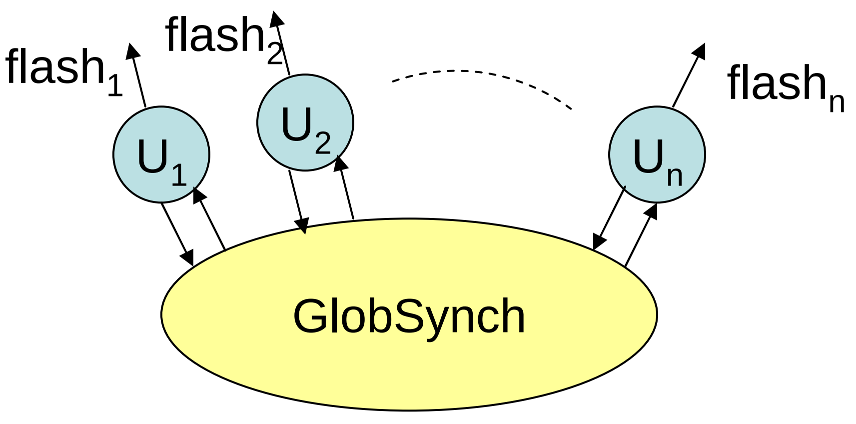
Example: Boolean matrix computation

- $n = m^3$ processes compute the transitive closure of $m \times m$ Boolean matrix M.
- $p_{i,j,k}$ repeatedly does:
  - read M(i,k), read M(k,j)
  - If both are 1 then write 1 in M(i,j)
- Each $flash_{i,j,k}$ in abstract session problem represents a chance for $p_{i,j,k}$ to read or write a matrix entry.
- With enough interleaving (O (log n) sessions), this is guaranteed to compute transitive closure.



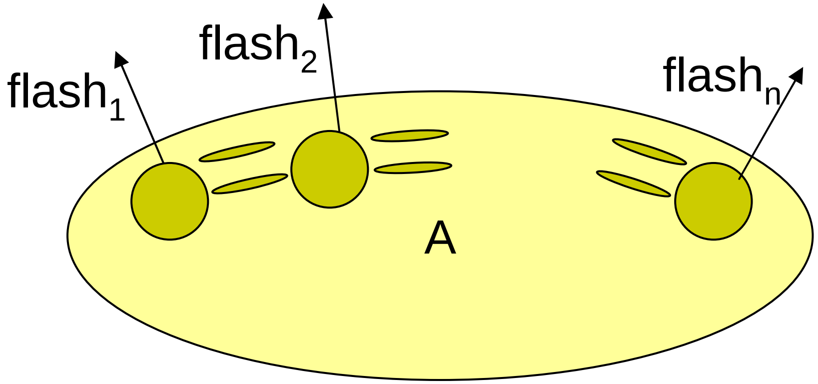
Synchronous solution

- Fast algorithm using GlobSynch:
  - Just flash once at every round.
  - k sessions done in time O(k d), assuming GlobSynch takes steps ASAP.



Asynchronous lower bound

- Consider distributed algorithm A that solves the k-session problem.
- Consists of process automata and FIFO send/receive channel automata.



- Assume:
  - d = upper bound on time to deliver any message (don't count pileups)
  - l = local processing time, l << d
- Define time measure T(A):
  - Timed execution α: Fair execution with times labeling events, subject to upper bound of d on message delay, l for local processing.
  - T(α) = time of last flash in α
  - T(A) = supremum, over all timed executions α, of T(α).

Lower bound

- Theorem 2: If A solves the k-session problem then $T(A) \geq (k-1)$ diam d.
- Factor of diam worse than the synchronous algorithm.

- Definition: Slow timed execution: All message deliveries take exactly the upper bound time d.

- Proof: By contradiction.
  - Suppose $T(A) < (k-1)$ diam d.
  - Fix α, any slow timed execution of A.
  - α contains at least k sessions.
  - α contains no flash event at a time $\geq (k-1)$ diam d.
  - So we can decompose $\alpha = \alpha_1 \underbrace{\alpha_2 \ldots \alpha_{k-1}\, \alpha''}_{\alpha'}$, where:
    - Time of last event in α' is $< (k-1)$ diam d.
    - No flash events occur in α''.
    - Difference between the times of the first and last events in each α_r is $<$ diam d.

Lower bound, cont'd

- Now reorder events in α, while preserving dependencies:
  - Events of same process.
  - Send and corresponding receive.
- Reordered execution will have < k sessions, a contradiction.
- Fix processes, j_0 and j_1, with $dist(j_0,j_1) = diam$ (maximum distance apart).
- Reorder within each α_r separately:
  - For α_1: Reorder to $\beta_1 = \gamma_1 \delta_{1,}$ where:
    - γ_1 contains no event of j_0, and
    - δ_1 contains no event of j_1.
  - For α_2: Reorder to $\beta_2 = \gamma_2 \delta_{2,}$ where:
    - γ_1 contains no event of j_1, and
    - δ_1 contains no event of j_0.
  - And alternate thereafter.

Lower bound, cont'd

- If the reordering yields a fair execution of A (can ignore timing), then we get a contradiction, because it contains $\leq$ k-1 sessions:
  - No session entirely within γ_1, (no event of j_0).
  - No session entirely within $\delta_1 \gamma_2$ (no event of j_1).
  - No session entirely within $\delta_2 \gamma_3$ (no event of j_0).
  - …
  - Thus, every session must span some γ_r - δ_r boundary.
  - But, there are only k-1 such boundaries.

- So, it remains only to construct the reordering.

Constructing the reordering

- WLOG, consider α_r for r odd.
- Need $\beta_r = \gamma_r \delta_r$, where γ_r contains no event of j_0, δ_r no event of j_1.

- If α_r contains no event of j_0 then don't reorder, just define $\gamma_r = \alpha_r$, $\delta_r = \lambda$.
- Similarly if α_r contains no event of j_1.
- So assume α_r contains at least one event of each.
- Let π be the first event of j_0, φ the last event of j_1 in α_r.

- Claim: φ does not depend on π.
- Why: Insufficient time for messages to travel from j_0 to j_1:
  - Execution α is slow (message deliveries take time d).
  - Time between π and φ is < diam d.
  - j_0 and j_1 are diam apart.

- Then, we can reorder α_r to β_r, in which π comes after φ.
- Consequently, in β_r, all events of j_1 precede all events of j_0.
- Define γ_r to be the part ending with φ, δ_r the rest.

Next time…

- Time, clocks, and the ordering of events in a distributed system.
- State-machine simulation.
- Vector timestamps.
- Reading:
  - Chapter 18
  - [Lamport time, clocks…paper]
  - [Mattern paper]

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