6.854J / 18.415J Advanced Algorithms
Fall 2008

18.415/6.854 Advanced Algorithms	November 19, 1992
Lecture 19	
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1 Lattices

Starting with today's lecture, we will look at problems involving lattices and algorithms for basis reduction of lattices. Applications of this topic include factoring polynomials, breaking cryptosystems, rounding an interior point to an optimal vertex in linear programming, and solving integer programs. We start with definitions:

Definition 1 *Given a set of vectors $b_1, \ldots, b_m \in \mathbb{Q}^n$, we define the lattice $L = L(b_1, \ldots, b_m) = \{\sum_{i=1}^m \lambda_i b_i : \lambda_i \in \mathbb{Z}\}$. Thus, L is the set of integral combinations of the vectors b_i.*

Example: $b_1 = (1,2), b_2 = (2,1), n = m = 2$.

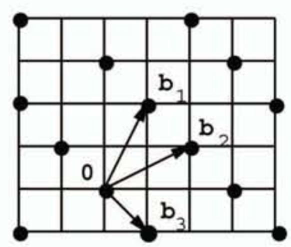

Figure 1: The lattice $L(b_1, b_2) = L(b_2, b_3)$

The simplest lattice is defined by unit vectors; $L(e_1, \ldots, e_n) = \mathbb{Z}^n$.

Definition 2 *A set of vectors $(b_1, \ldots, b_m)$ is a basis for L if $b_1, \ldots, b_m$ are linearly independent (with respect to $\mathbb{Z}$) and $L = L(b_1, \ldots, b_m)$.*

Every lattice has a basis, and its dimension is fixed. A given lattice can have many bases. In the above example for instance, Figure 1 shows that $L(b_1, b_2) = L(b_2, b_3)$. This follows from the fact that $b_3 \in L(b_1, b_2)$ and $b_1 \in L(b_2, b_3)$. The basic operation to obtain another basis for a lattice is to subtract from one of the vectors an integral combination of the others. This idea is presented in our first claim:

Claim 1 $L(b_1, \ldots, b_m) = L(b_1, \ldots, b_{m-1}, b_m - \sum_{i=1}^{m-1} \alpha_i b_i)$ *for $\alpha_i \in \mathbb{Z}$.*

[1]These notes are based on last year's class notes, prepared by Atul Shrivastava and by David Gupta.

Proof: Let $x \in L(b_1, \ldots, b_m)$. Then,

$$x = \sum_{i=1}^{m} \lambda_i b_i = \sum_{i=1}^{m-1} (\lambda_i + \alpha_i \lambda_m) b_i + \lambda_m (b_m - \sum_{i=1}^{m-1} \alpha_i b_i))$$

Since $(\lambda_i + \alpha_i \lambda_m) \in \mathbb{Z}$, we have $x \in L(b_1, \ldots, b_{m-1}, b_m - \sum_{i=1}^{m-1} \alpha_i b_i)$.
Now let $x \in L(b_1, \ldots, b_{m-1}, b_m - \sum_{i=1}^{m-1} \alpha_i b_i)$. Then

$$x = \sum_{i=1}^{m-1} \beta_i b_i + \beta_m (b_m - \sum_{i=1}^{m-1} \alpha_i b_i) = \sum_{i=1}^{m} \lambda_i b_i,$$

where $\lambda_i = (\beta_i - \alpha_i \beta_m)$ for $i = 1, \ldots, m-1$ and $\lambda_m = \beta_m$. □

Definition 3 *L is a full lattice in $\mathbb{Q}^n$ if it can be generated by n linearly independent vectors.*

Example: $L((0,1),(0,3))$ is not a full lattice in $\mathbb{Q}^2$.

Theorem 2 below implies that any one-dimensional lattice has a basis with at most one vector. In the problem set we will show that any lattice in $\mathbb{Q}^n$ has a basis with at most n vectors.

A given lattice can be reduced to a full lattice in polynomial time by restricting our attention to the affine space spanned by the vectors defining the lattice. As a result, without loss of generality, we will look only at lattices that are full.

Also, we will see (as an exercise in the last problem set) that, given a set of vectors $b_1, \ldots, b_m$, a basis for the lattice $L(b_1, \ldots, b_m)$ can be computed in polynomial time. Therefore, without loss of generality, we shall always assume that we are given a full lattice and a basis of that lattice.

Let us show how to compute a basis of L in polynomial time in the case $n = 1$. The general case can be solved in a recursive manner using the result for $n = 1$ (see the problem set). We are thus given m integers $b_1, \ldots, b_m$, and we would like to find an integer a such that $L(b_1, \ldots, b_m) = L(a)$.

Theorem 2 *Let $b_1, \ldots, b_m \in \mathbb{N}$. Then $L(b_1, \ldots, b_m) = L(\gcd(b_1, \ldots, b_m))$.*

Proof: The case $m = 1$ is trivial. Consider the case $m = 2$. Assume w.l.o.g. that $0 \leq b_1 \leq b_2$. We prove that $L(b_1, b_2) = L(\gcd(b_1, b_2))$ by induction on b_1.
If $b_1 = 0$ then $L(b_1, b_2) = L(b_2) = L(\gcd(b_1, b_2))$. If $b_1 > 0$, then

$$\begin{aligned} L(b_1, b_2) &= L(b_2 - b_1 \lfloor b_2/b_1 \rfloor, b_1) \text{ by Theorem 1 above} \\ &= L(\gcd(b_1, b_2 - b_1 \lfloor b_2/b_1 \rfloor)) \text{ by the induction hypothesis} \\ &= L(\gcd(b_1, b_2)) \text{ by Euclid's algorithm} \end{aligned}$$

The case $m > 2$ is shown by induction on m. Assume the theorem is true for m. Then

$$\begin{aligned} L(b_1, \dots, b_m, b_{m+1}) &= L(L(b_1, \dots, b_m), b_{m+1}) \\ &= L(L(\gcd(b_1, \dots, b_m)), b_{m+1}) \\ &= L(\gcd(b_1, \dots, b_m), b_{m+1}) \\ &= L(\gcd(b_1, \dots, b_m, b_{m+1})) \end{aligned}$$

□

Note that the greatest common divisor of two integers can be calculated in polynomial time by Euclid's algorithm, since every two steps reduce the bit size of the maximum by at least 1. By applying Euclid's algorithm repeatedly, we can calculate the GCD of several integers in polynomial time.

2 Combinatorial Application

Suppose we are given a graph $G = (V, E)$, and we want to assign colors to the edges such that no vertex is covered by two edges of the same color, and the number of colors is minimized. The minimum number of colors is at least $d_{\max}$, the maximum degree of any node. Vizing showed that the minimum number of colors is at most $d_{\max}+1$. However, deciding whether the minimum number of colors is $d_{\max}$ or $d_{\max}+1$ is NP-hard, even for special subclasses of graphs.

Consider the class of cubic graphs, the graphs for which every vertex has degree 3. Deciding whether the mininum number of colors needed is $d_{\max} = 3$ or 4 is NP-hard. But if there is a three-coloring, then the edges of the same color make a perfect matching. So there is a three-coloring if and only if there is a partition of E into perfect matchings.

We can identify the perfect matchings M with vectors b in $\mathbb{Z}^{|E|}$. If $e \in M$ then $b_e = 1$, otherwise $b_e = 0$. Let L be the lattice spanned by these vectors. If there is a three-coloring, then $(1, 1, \dots, 1) \in L$. The converse isn't necessarily true, but this does give us a way to show that a graph is not 3-colorable if we can show that $(1, \dots, 1)$ is not in the lattice.

3 Shortest Lattice Vector Problem (SLVP)

Given n linearly independent vectors $b_1, \dots, b_n$ in $\mathbb{Q}^n$ (remember that we can assume w.l.o.g. that we are given a basis of a full lattice), we want to find a nonzero vector $a \in L(b_1, \dots, b_n)$ such that $\|a\| = \sqrt{a \cdot a}$ is minimized. This problem is called *the shortest lattice vector problem*. Let $\Lambda(L) = \min_{a \in L, a \neq 0} \|a\|$.

The shortest lattice vector problem (SLVP) is believed to be NP-hard. If $\| \ \|$ is replaced by $\| \ \|_\infty$, then it is known to be NP-hard (Van Emde Boas 1981). However, if n is fixed, the SLVP problem is solvable in polynomial time. We will treat below the cases $n = 1$ or 2.

For $n = 1$, the case is trivial since a is a shortest lattice vector in $L(a)$.

Let us now treat the case $n = 2$. We shall find a basis $(b_1, b_2) \in \mathbb{Q}^2 \times \mathbb{Q}^2$ of L in polynomial time in which b_1 is a shortest non-zero lattice vector. We use the $60°$-algorithm due to Gauss (1801).

> **If** $\|b_1\| > \|b_2\|$ **then** swap b_1, b_2
> **Repeat** Choose $m \in \mathbb{Z}$ to minimize $\|b_2 - mb_1\|$
> $\qquad b_2 := b_2 - mb_1$.
> **Until** $m = 0$
> **Return** b_1, b_2

Claim 3 *The $60°$-algorithm terminates in polynomial time.*

The proof is analogous to the proof that Euclid's algorithm terminates in polynomial time (see problem set). As in Euclid's algorithm, the number of iterations is logarithmic in the numbers. The reasons are similar, but more complicated.

Theorem 4 *The $60°$-algorithm returns a shortest non-zero vector in L.*

Proof: At termination, we have

- $\|b_1\| \le \|b_2\|$
- $\|b_2\| \le \|b_2 - \mu b_1\|$, for all $\mu \in \mathbb{Z}$.

Since $\|b_2\| \le \|b_2 - \mu b_1\|$ for any integer μ, the orthogonal projection of b_2 on b_1 is between $b_1/2$ and $-b_1/2$ (see Figure 2). On the other hand, $\|b_2\| \ge \|b_1\|$ and so b_2 is outside the circle $(0, \|b_1\|)$. This implies that $|\cos\alpha| \le 1/2$ and so $60° \le \alpha \le 120°$. In fact, because of the hexagonal lattice, this bound is tight.

Let $a = \lambda_1 b_1 + \lambda_2 b_2$ be a shortest non-zero vector in L. Since $\alpha \ge 60°$ and $\alpha + \beta + \gamma = 180°$, we have $\beta \le \alpha$ or $\gamma \le \alpha$ (see Figure 3). Therefore we have $\|a\| \ge |\lambda_1|\|b_1\|$ or $\|a\| \ge |\lambda_2|\|b_2\|$, since the length of the sides of a triangle are in the same order as the angles they face. Since the λ_i's are integers and $\|b_1\| \le \|b_2\|$, this implies that $\|b_1\| \le \|a\|$. $\square$

Since $60° \le \alpha \le 120°$, b_1 and b_2 are almost orthogonal. One can prove that (b_1, b_2) is a couple of independent vectors in L that

1. maximizes $\sin\alpha$
2. minimizes $\|b_1\| \ \|b_2\|$

In fact, we will see that these two statements are equivalent.

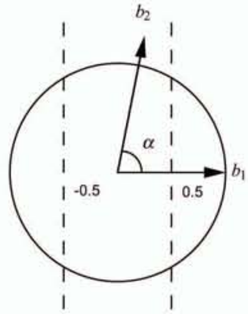

Figure 2: $60° \leq \alpha \leq 120°$.

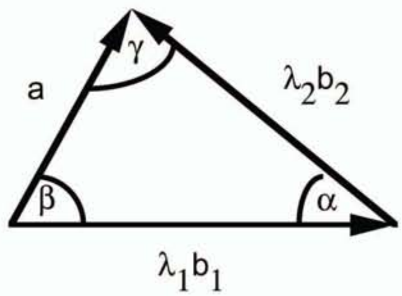

Figure 3: $\beta \leq \alpha$ or $\gamma \leq \alpha$.

4 Minimum Basis Problem

Given a basis $(b_1, \ldots, b_n)$ of a full lattice $L \subset \mathbb{Q}^n$, consider the non-singular $n \times n$ matrix $B = [b_1 \ldots b_n]$. We know that $|\det(B)|$ is the volume of the parallelepiped defined by $b_1, \ldots, b_n$.

Theorem 5 *Given a full lattice L, $|\det(B)|$ is independent of B, for any basis B of L.*

Proof: Let B and B' be two bases of L. For $1 \leq i \leq n$, we have $b'_i = \sum_{j=1}^{n} \lambda_{ij} b_j$, where the λ_{ij} are integers. In other words, $B' = BP$, where P is an integral $n \times n$ matrix. Therefore, $|\det B'| = |\det B||\det P|$. But $|\det B'| \neq 0$ since B' is non-singular. Hence $|\det P| \neq 0$ and so $|\det P| \geq 1$ since P is integral. This implies that $|\det B'| \geq |\det B|$. By symmetry, it follows that $|\det B'| = |\det B|$. □

Since $|\det(B)|$ does not depend on the choice of the basis for a given lattice L, let $\det(L) = |\det(B)|$. When $n = 2$, we have $|\det(B)| = \|b_1\| \, \|b_2\| \sin\alpha$, and so minimizing $\|b_1\| \, \|b_2\|$ is equivalent to maximize $\sin\alpha$.

From linear algebra, we know that it is easier to deal with bases which are orthogonal. However, in the case of lattices, this is not always possible. Nevertheless, we are interested in finding a basis that is "somewhat orthogonal". The case for $n = 2$ treated above and Theorem 5 therefore motivates the following problem, called the *minimum basis problem*.

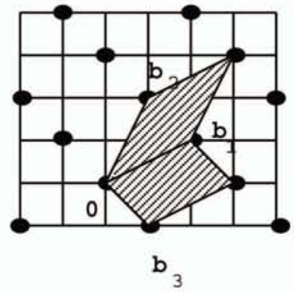

Figure 4: The determinant of a basis is constant in absolute value.

> Given a lattice L, we want to find a basis $(b_1, \ldots, b_n)$ that minimizes the product $\|b_1\| \cdots \|b_n\|$.

This problem turns out to be NP-hard (Lovász). However, there are α- approximation algorithms for this problem where α depends only on the dimension of the basis of the lattice. Fortunately, there is a general lower bound on the size of a minimum basis (which is attained by some lattices and is thus tight) and, in any *given* dimension, *all* lattices have a basis whose size is at most a constant multiple of the general lower bound. This will allow us to develop an approximation algorithm.

Claim 6 *(Hadamard's Inequality). For any basis of L,* $\det L \leq \|b_1\| \ldots \|b_n\|$.

Theorem 7 *(Hermite 1850) For any dimension n, there is a constant c_n such that for any lattice $L \in \mathbb{Q}^n$ there is a basis $b_1, \ldots, b_n$ of L such that $\|b_1\| \ldots \|b_n\| \leq c_n \det L$.*

We will, in fact, take c_n to be the smallest such constant. So what is c_n? In one dimension, a given lattice has only two bases, which are the two minimum nonzero vectors. These vectors are both exactly the size of the lattice spacing, which is also the determinant of the lattice. Then in one dimension, a minimum basis always has a ratio of exactly 1, so $c_1 = 1$. In two dimensions, we know from the analysis of Gauss's algorithm that the angle, α, between the vectors in a minimum basis is at least 60°. But for basis vectors b_1, b_2, we know that $\det L = \|b_1\|\|b_2\| \sin\alpha \geq \|b_1\|\|b_2\| \sin 60° = \|b_1\|\|b_2\|\frac{\sqrt{3}}{2}$. So $c_2 = \frac{2}{\sqrt{3}}$.

In 1850, Hermite proved that c_n could be bounded by $2^{O(n^2)}$.

In 1984, Schnorr proved that c_n was bounded by n^n.

Unfortunately, neither of the above proofs is algorithmic because neither one gives us any insight on how to actually go about computing a small basis.

In 1983, however, Lenstra, Lenstra, and Lovász provided an algorithm that produces a reduced basis whose size is at most $2^{O(n^2)} \det L$. This algorithm can also be used to approximate the shortest lattice vector problem.

5 More on the Shortest Lattice Vector Problem

Definition 4 *A body, K, is said to be symmetric with respect to the origin if $x \in K \Rightarrow -x \in K$. Note that this statement is its own inverse, so we can think of K being symmetric with respect to the origin as meaning $x \in K \Leftrightarrow -x \in K$.*

We present Minkowski's Theorem without proof as background for a useful corollary.

Theorem 8 *(Minkowski's Theorem-1891) Let K be a convex body symmetric with respect to the origin and let lattice $L \in \mathbb{Q}^n$ be such that $Vol(K) \geq 2^n \det L$. Then K contains a nonzero lattice point.*

Corollary 9 *Consider the norm $\|\cdot\|_p$ for any integer p. Then there is a nonzero $a \in L$ such that $\|a\|_p \leq 2(\frac{\det L}{v_p})^{1/n}$ where $v_p = Vol(\{x : \|x\|_p \leq 1\})$.*

Example: $p = \infty$; $v_\infty = 2^n$. Then there is a nonzero $a \in L$ such that $\|a\|_\infty \leq 2(\frac{\det L}{2^n})^{1/n}$. Thus $(\max_j |a_j|)^n \leq \det L$.

We can give a proof by picture for Corollary 9 when $p = \infty$. Let

$$t = \min_{\substack{a \neq 0 \\ a \in L}} (\max_j |a_j|),$$

i.e. t is the smallest nonzero $\| \ \|_\infty$ norm of lattice vectors. We place a cube with edge length t centered on each lattice point. The cubes may touch, but they don't overlap. The volume per lattice element is t^n. On the other hand, we can cover the space with parallelepipeds of volume $\det L$, one per lattice point. Thus we get the inequality $t^n \leq \det L$.

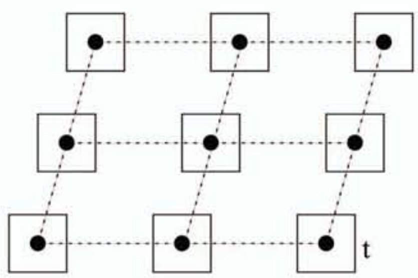

Figure 5: Covering the lattice with cubes and parallelepipeds.

Next time we will discuss L[3], the Lenstra-Lenstra-Lovász theorem. We will give the algorithm and applications.

1 Gram-Schmidt Decomposition

First, recall 3 facts about full lattices L and their bases from last lecture:

1. Hadamard's Inequality: If $b_1, \ldots, b_n$ is a basis for L, then $|detL| \leq ||b_1|| \cdots ||b_n||$.
2. Hermite's Thm: $\forall n > 0$, there is a constant c_n such that for any lattice $L \subset \mathbb{Q}^n$, there is a basis $b_1, \ldots, b_n$ of L such that $||b_1|| \cdots ||b_n|| \leq c_n detL$.
3. Corollary to Minkowski's Theorem: For any lattice $L \subset \mathbb{Q}^n$, there is a nonzero vector $a \in L$ such that $||a||_\infty \leq (detL)^{1/n}$ (and hence, such that $||a||_2 \leq \sqrt{n}(detL)^{1/n}$).

We remark that the corollary to Minkowski's Theorem is as good as one can achieve, asymptotically, in the sense that for any $n > 0$, there is a lattice $L \subset \mathbb{Q}^n$ such that $\Lambda(L) \geq \sqrt{\frac{1}{e\pi}}\sqrt{n}(detL)^{1/n}$.

Now remember the Gram-Schmidt Decomposition:

Given the vectors $b_1, \ldots, b_m$, the following procedure calculates m *orthogonal* vectors $b_1^*, \ldots, b_m^*$ such that b_i^* is the projection of b_i onto the space orthogonal to $b_1, \ldots, b_{i-1}$:

$$\begin{aligned} b_1^* &= b_1 \\ b_i^* &= b_i - \sum_{j=1}^{i-1} \frac{(b_i, b_j^*)}{(b_j^*, b_j^*)} b_j^* \end{aligned}$$

The vectors b_i^* are not necessarily in the lattice because the coefficients $\mu_{ij} = (b_i, b_j^*)/(b_j^*, b_j^*)$ are not necessarily integral. We can write $b_i = \sum_{j=1}^{i} \mu_{ij} b_j^*$ where $\mu_{ii} = 1$. Equivalently, $B = B^* P$ where

$$P = \begin{pmatrix} 1 & \mu_{21} & \cdots & \mu_{n,1} \\ 0 & 1 & \ddots & \mu_{n,2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix}$$

Notice that $det(B) = det(B^*)det(P) = det(B^*)$, since all lower triangular entries of P are zero.

Claim 1 $\Lambda(L) \geq \min_i ||b_i^*||$ *for any basis* $(b_1, \ldots, b_n)$ *of* L.

[1]These notes are based largely on notes from Atul Shrivastava, David Gupta, Marcos Kiwi, Andrew Sutherland, and Ethan Wolf.

Most of the proofs of this lecture were omitted in class, but are included for completeness.
Proof: Let $a \in L$ be a minimum-length lattice vector: $\|a\|_2 = \Lambda(L)$. Since $a \in L$, then we can write a as $\sum_{i=1}^{n} \lambda_i b_i, \lambda_i \in \mathbb{Z}$. Let k be the last index for which $\lambda_k \neq 0$. Then $\lambda_j = 0$ for all $j > k$. By substituting in from Gram-Schmidt orthogonalization, we get

$$a = \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{i} \mu_{i,j} b_j^* = \sum_{j=1}^{n} \sum_{i=j}^{n} \lambda_i \mu_{i,j} b_j^*.$$

Let us define λ_j^* for $1 \leq j \leq n$ by $\lambda_j^* \equiv \sum_{i=j}^{n} \lambda_i \mu_{i,j}$. Then $a = \sum_{j=1}^{n} \lambda_j^* b_j^*$. Since the b_j^*'s are orthogonal to each other, we have that

$$\|a\|_2^2 = \sum_{j=1}^{n} (\lambda_j^*)^2 \|b_j^*\|^2 \geq (\lambda_k^*)^2 \|b_k^*\|^2.$$

Thus $\|a\|_2 \geq |\lambda_k^*| \|b_k^*\|$. Note that $\lambda_k^* = \lambda_k \mu_{k,k} + \lambda_{k+1}\mu_{k+1,k} + \ldots = \lambda_k$. Thus $\|a\|_2 \geq |\lambda_k| \|b_k^*\|$. Since $\lambda_k \in Z$ and $\lambda_k \neq 0$, then $|\lambda_k| \geq 1$. So $\|a\|_2 \geq \|b_k^*\| \geq \min_i \|b_i^*\|$. Thus $\Lambda(L) \geq \min_i \|b_i^*\|$. □

2 Lovasz-reduced Bases

In Gauss' algorithm, we were performing swaps to insure that the basis satisfies certain properties. In general, to insure that the first vector of the basis is reasonably short, we shall impose that the switching of any b_i with b_{i+1} does not decrease $\|b_i^*\|$ (recall that the Gram-Schmidt orthogonalization depends upon the ordering of the vectors). This requirement can be more easily stated by using the following observation.

Claim 2 *Let $(b_1, \ldots, b_n)$ be a basis for lattice L. If we switch b_i with b_{i+1} to produce the new basis $(c_1, \ldots, c_n)$, then $b_j^* = c_j^*$ for $j \neq i, i+1$ and $c_i^* = b_{i+1}^* + \mu_{i+1,i} b_i^*$.*

Proof: From Gram-Schmidt orthogonalization, c_j^* is the component of c_j orthogonal to the span of $\{c_1, c_2, \ldots, c_{j-1}\}$, but this set is the same as $\{b_1, b_2, \ldots, b_{j-1}\}$ for $j \neq i+1$. Since $c_i = b_i$ for $j \neq i, i+1$, we get that $b_j^* = c_j^*$ for $j \neq i, i+1$. We also have that c_i^* is the component of $c_i = b_{i+1}$ orthogonal to the span of $\{b_1, b_2, \ldots, b_{i-1}\}$. From the original Gram-Schmidt orthogonalization, we know that $b_{i+1} = \sum_{k=1}^{i+1} \mu_{i+1,k} b_k^*$, so $c_i = \sum_{k=1}^{i-1} \mu_{i+1,k} b_k^* + b_{i+1}^* + \mu_{i+1,i} b_i^*$. Removing the component of each side in the span of $\{b_1, b_2, \ldots, b_{i-1}\}$, we get $c_i^* = b_{i+1}^* + \mu_{i+1,i} b_i^*$. □

This claim says that we can require that switching neighboring basis vectors not help reduce small $\|b_i^*\|$'s by requiring that $\|b_{i+1}^* + \mu_{i+1,i} b_i^*\|^2 \geq \|b_i^*\|^2$ for $1 \leq i < n$.

We would also like for our basis vectors to be as close to orthogonal as possible. If they were strictly orthogonal, then each $\mu_{i,j}$ would be zero. But this is not possible for most lattices. We would like to require that $|\mu_{i,j}|$ be as small as possible for each i, j. We now present a form of these two requirements sufficiently loose enough to guarantee the existence of such a basis and to allow for a polynomial-time algorithm.

Definition 1 *A Lovasz-reduced basis for L is a basis $(b_1, \ldots, b_n)$ for which*

1. $|\mu_{i,j}| \leq \frac{1}{2}$ *for* $1 \leq j < i \leq n$.

2. $\|b^*_{i+1} + \mu_{i+1,i} b^*_i\|^2 \geq \frac{3}{4}\|b^*_i\|^2$ *for* $1 \leq i < n$.

Proposition 3 *Let* $(b_1, \ldots, b_n)$ *be a Lovasz-reduced basis of a lattice* L. *Then*

1. $\|b_1\| \leq 2^{\frac{n-1}{4}}(\det L)^{\frac{1}{n}}$.

2. $\|b_1\| \leq 2^{\frac{n-1}{2}} \min_i \|b^*_i\| \leq 2^{\frac{n-1}{2}} \Lambda(L)$.

3. $\|b_1\| \ldots \|b_n\| \leq 2^{\frac{1}{2}\binom{n}{2}} \det L$.

Proof of Proposition 3:

Claim 4 $\|b_1\|^2 \leq 2^{j-1}\|b^*_j\|^2, 1 \leq j \leq n$.

From property (ii) of a Lovasz-reduced basis and the orthogonality of the b^*_i's, we have

$$\frac{3}{4}\|b^*_i\|^2 \leq \|b^*_{i+1} + \mu_{i+1,i} b^*_i\|^2 = \|b^*_{i+1}\|^2 + \mu^2_{i+1,i}\|b^*_i\|^2.$$

Since from property (i), we know that $\mu^2_{i+1,i} \leq \frac{1}{4}$, we have that

$$\frac{3}{4}\|b^*_i\|^2 \leq \|b^*_{i+1}\|^2 + \frac{1}{4}\|b^*_i\|^2$$

or

$$\|b^*_i\|^2 \leq 2\|b^*_{i+1}\|^2.$$

Repeatedly substituting into the above starting with $i = 1$, we obtain

$$\|b^*_1\|^2 \leq 2^{j-1}\|b^*_j\|^2, 1 \leq j \leq n.$$

Since $b^*_1 = b_1$, this becomes $\|b_1\|^2 \leq 2^{j-1}\|b^*_j\|^2, 1 \leq j \leq n$, which proves the claim. $\diamond$

Solving the above for $\|b^*_j\|^2$, we can square both sides of the definition of $\det L$ and perform a substitution to obtain

$$(\det L)^2 = \prod_{j=1}^{n} \|b^*_j\|^2 \geq \prod_{j=1}^{n} 2^{1-j}\|b_1\|^2 = 2^{\frac{-n(n-1)}{2}}\|b_1\|^{2n}.$$

Raising both sides to the power $\frac{1}{2n}$ gives

$$(\det L)^{\frac{1}{n}} \geq 2^{\frac{-(n-1)}{4}}\|b_1\|$$

$$\|b_1\| \leq 2^{\frac{n-1}{4}}(\det L)^{\frac{1}{n}}.$$

This proves part 1 of the proposition.

Let k be the index for which $\min_i \|b^*_i\|$ is attained, so that $\|b^*_k\| = \min_i \|b^*_i\|$. Then by the above claim: $\|b_1\|^2 \leq 2^{k-1}\|b^*_k\|^2 \leq 2^{n-1}\|b^*_k\|^2 = 2^{n-1}\min_i \|b^*_i\|^2$. Taking the square root of both sides: $\|b_1\| \leq 2^{\frac{n-1}{2}} \min_i \|b^*_i\|$. By applying Claim 1, we can extend this result to: $\|b_1\| \leq 2^{\frac{n-1}{2}}\Lambda(L)$, which is the statement of part 2 of the proposition.

Recall that we have $b_i = \sum_{j=1}^{i} \mu_{i,j} b_j^*$ by Gram-Schmidt orthogonalization. It also follows from the proof of Claim 4, that $\|b_j^*\|^2 \leq 2^{i-j}\|b_i^*\|^2$ for $j < i$. Then making use of the orthogonality and the fact that the coefficients satisfy property (i) of a Lovasz-reduced basis, we get that

$$\|b_i\|^2 = \sum_{j=1}^{i} \mu_{i,j}^2 \|b_j^*\|^2 \leq \|b_i^*\|^2 + \sum_{j=1}^{i-1} \frac{1}{4} 2^{i-j} \|b_i^*\|^2 = \|b_i^*\|^2 (1 + \sum_{j=1}^{i-1} \frac{1}{4} 2^{i-j}) \leq 2^{i-1} \|b_i^*\|^2.$$

Multiplying these inequalities for all values of i gives

$$\|b_1\|^2 \ldots \|b_n\|^2 \leq \prod_{i=1}^{n} 2^{i-1} \|b_i^*\|^2 = (\prod_{i=1}^{n} 2^{i-1})(\prod_{i=1}^{n} \|b_i^*\|^2) = 2^{\frac{n(n-1)}{2}} (\det L)^2 = 2^{\binom{n}{2}} (\det L)^2.$$

Taking the square root of both sides gives

$$\|b_1\| \ldots \|b_n\| \leq 2^{\frac{1}{2}\binom{n}{2}} \det L.$$

This proves part 3 of the proposition. □

We now present an algorithm due to A. K. Lenstra, H. W. Lenstra and L. Lovász [2] which computes a reduced basis in polynomial time. We assume throughout that that we are dealing with integral lattices; i.e., we assume that every basis consists of integral vectors.

3 Lenstra-Lenstra-Lovász (L^3) Basis Reduction Algorithm

The algorithm receives as input a set of linearly independent vectors $b_1, b_2, \ldots, b_n \in \mathbb{Z}^n$, and outputs a Lovász-Reduced Basis of $L(b_1, \ldots, b_n)$.

Initialization Find the Gram-Schmidt orthogonalization $(b_1^*, b_2^*, \ldots, b_n^*)$ of $(b_1, b_2, \ldots, b_n)$.

Step 1 Make sure that property 1. of a Lovász-Reduced Basis holds.

```
For i = 2 to n do
    For j = i − 1 down to 1 do
        b_i ← b_i − ⌈μ_ij⌋ b_j
        For k = 1 to j do
            μ_ik ← μ_ik − ⌈μ_ij⌋ μ_jk
        {Note that for k > j, b*_k ⊥ b_j so that μ_ik is unaffected}
```

Step 2 If there exists a i for which property 2 of a Lovász-Reduced Basis is not satisfied, swap b_i and b_{i+1}, update the b_k^*'s for $k = i, i+1$, update the μ_{kj}'s for $k = i, i+1$, and go to step 1.

Else return $(b_1, b_2, \ldots, b_n)$.

Note that

1. If the algorithm terminates, it returns a Lovász-Reduced Basis.

2. $b_1^*, b_2^*, \ldots, b_n^*$ are not affected in step 1, since $span\{b_1, b_2, \ldots, b_i\}$, for $i = 1 \ldots n$ is not modified performing this step.

3. After step 1 $|\mu_{ij}| \leq \frac{1}{2}$.

It is not clear that the algorithm makes any progress at each iteration. The following result shows that in fact L^3 terminates.

Theorem 5 *The L^3 algorithm terminates after $O(n^2 \log \beta)$ iterations, where $\beta = \max_i \|b_i^0\|$ (the superscript 0 denotes input vectors).*

Proof: Define a potential

$$\Phi(\bar{b}) = \prod_{j=1}^{n-1} \|b_j^*\|^{2(n-j)} = \prod_{j=1}^{n-1} [\prod_{i=1}^{j} \|b_i^*\|^2] = \prod_{j=1}^{n-1} \det D_j,$$

where $D_j = [d_j(k,l)] = [(b_k, b_l)]_{k,l \leq j}$. Hence, $\Phi(\bar{b})$ is a positive integer, since the D_j's are integral matrices.

In step 1, $\Phi(\bar{b})$ does not change because the b_i^*'s do not change.

In step 2, let $\bar{c} = (c_1, \ldots, c_n) = (b_1, b_2, \ldots, b_{i+1}, b_i, \ldots, b_n)$ be the new basis created after swapping b_i and b_{i+1}. Since c_j^* is the projection of c_j onto the orthogonal of $span\{c_1, c_2, \ldots, c_{j-1}\}$, it follows that $c_j^* = b_j^*$ for $j \neq i, i+1$. Furthermore, $c_i^* = b_{i+1}^* + \mu_{i+1,i} b_i^*$, since c_i^* is the projection of b_{i+1} onto the orthogonal of $span\{b_1, \ldots, b_{i-1}\}$ and b_{i+1}^* is the projection of b_{i+1} onto the orthogonal of $span\{b_1, \ldots, b_i\}$.
Moreover, since $\|b_1^*\| \ldots \|b_n^*\| = \det(L) = \|c_1^*\| \ldots \|c_n^*\|$ we have that $\|b_i^*\| \|b_{i+1}^*\| = \|c_i^*\| \|c_{i+1}^*\|$.

Thus,

$$\frac{\Phi(\bar{c})}{\Phi(\bar{b})} = \frac{\|c_i^*\|^{2(n-i)} \|c_{i+1}^*\|^{2(n-i-1)}}{\|b_i^*\|^{2(n-i)} \|b_{i+1}^*\|^{2(n-i-1)}} = \frac{\|c_i^*\|^2}{\|b_i^*\|^2} < \frac{3}{4}$$

Consequently, the number of iterations of the L^3 algorithm is at most $\frac{\log \Phi_0}{\log 4 - \log 3}$, where Φ_0 is the initial value of Φ.
Now let $\bar{b}^0 = \{b_1^0, b_2^0, \ldots, b_n^0\}$ be the basis given as input. Then $\Phi_0 = \Pi_{j=1}^{n-1} \|(b_j^0)^*\|^{2(n-j)}$. But $\|(b_j^0)^*\| \leq \|b_j^0\|$, thus $\Phi_0 \leq \Pi_{j=1}^{n-1} \|b_j^0\|^{2(n-j)} \leq \beta^{n(n-1)}$, implying that $\log \Phi_0 \leq n(n-1) \log \beta$. It follows that the algorithm terminates after executing step 2 at most $\frac{n(n-1)\log \beta}{\log 4 - \log 3} = O(n^2 \log \beta)$ times. □
(Note that in the proof above the number 3/4 used in condition (ii) could be replaced by $1 - \epsilon$ for any $\epsilon > 0$ and the theorem would still hold).

Corollary 6 *The L^3 algorithm performs $O(n^5 \log \beta)$ arithmetic operations.*

The issue of how large the b_i's can become during the L^3 algorithm was not covered in class. The proof that at any time $size(b_i)$ remains polynomially bounded can be seen in last year lecture notes. In fact it can be shown that at any time $\|b_i\| \leq (1+2\beta^{2n+1}\sqrt{n})^n\beta\sqrt{n}$. This result completes the proof that the L^3 algorithm runs in polynomial time.

Beginning this lecture we shall be studying applications of these results in cryptography and simultaneous diophantine approximation. Other applications of the results we have seen relate to polynomial-time integer linear programming for programs of fixed dimension and polynomial-time factorization of polynomials over the rationals.

4 Diophantine Approximation

In a general sense, the Diophantine approximation problem is about how to "round" a number $\alpha \in \mathbb{R}$, meaning that we replace it by a rational number which is of a sufficiently simple form and at the same time sufficiently close to α. If we prescribe the denominator to q of this rational number p/q, then the best choice for p is $\lceil \alpha q \rfloor$. The error resulting from such a rounding is

$$|\alpha - \frac{p}{q}| \leq \frac{1}{2q}.$$

We shall find, however, that often this approximation is not good enough. A classical result of Dirichlet says that if we do not prescribe the denominator, but only an upper bound M for it, then there always exists a rational number p/q such that

$$|\alpha - \frac{p}{q}| \leq \frac{1}{Mq}, \quad 0 < q \leq M.$$

There also exists a classical method to find such a rational number p/q: this is the so-called *continued fraction expansion* of α. For an irrational number α, this expansion is infinite; for a rational number α, it is finite and of polynomial length.

Khintchine (1956) even showed that continued fractions can be used to solve the following best approximation problem.

> Given $\alpha \in \mathbb{Q}$ (or $\in \mathbb{R}$) and an integer $M > 0$, find a rational p/q with $0 < q \leq M$ such that $|\alpha - p/q|$ is as small as possible.

This often produces very good approximations. For example, if $\alpha = \pi$ and $M = 150$ the best approximation we can obtain using $q \leq 150$ is $355/113 = 3.1415929$.

5 Simultaneous Diophantine Approximation (SDA)

Suppose now we wish to approximate several values at once. i.e. we are given $M = 100$ and we wish to approximate $\alpha_1 = .1428, \alpha_2 = .2213, \alpha_3 = .6359$. Note that $\alpha_1 + \alpha_2 + \alpha_3 = 1$. If we approximate each value separately, we find that $\frac{p_1}{q_1} = \frac{1}{7} = .1428\cdots, \frac{p_2}{q_2} = \frac{2}{9} = .2222\cdots, \frac{p_3}{q_3} = \frac{7}{11} = .6363\cdots$. Unfortunately, $\frac{p_1}{q_1} + \frac{p_2}{q_2} + \frac{p_3}{q_3} \neq 1$. Thus, as a group these approximations are not good, since we would like our approximations to mantain "simple" equalities relating the α_i's. This is known as the SDA problem and is stated as follows:

> Given $\alpha_1, \cdots, \alpha_n \in \mathbb{Q}$, integer $M > 0$, and $0 < \epsilon < 1$, find $p_1, \cdots, p_n, q \in \mathbb{Z}$ s.t. $0 < q \leq M, |q\alpha_i - p_i| \leq \epsilon$ for all i. (Note that $|q\alpha_i - p_i| \leq \epsilon$ is equivalent to $|\alpha_i - \frac{p_i}{q}| \leq \frac{\epsilon}{q}$).

An equivalent statement of the problem is: given $0 < \epsilon < 1$, $M > 0$, and $\alpha = (\alpha_1, \cdots, \alpha_n)^T$ find $y = (\frac{p_1}{q}, \cdots, \frac{p_n}{q})^T$ such that $\|\alpha - y\|_\infty \leq \frac{\epsilon}{q}$. Now, if ϵ is too small, p_i and q may not exist. So we can look at this as a decision problem. Unfortunately, this decision problem has been shown to be NP-complete by Lagarias [1]. It has been shown, however, that for ϵ sufficiently large, a solution always exists.

Theorem 7 *(Dirichlet)* SDA *has a solution if* $M \geq \epsilon^{-n}$.

Proof: Define a lattice $L \subseteq \mathbb{Q}^{n+1}$ by $L = L(b_0, \ldots, b_n)$ where

$$
\begin{aligned}
b_0 &= (\alpha_1, \cdots, \alpha_n, \delta)^T \\
b_1 &= (-1, 0, \cdots, 0)^T = -e_1^T \\
&\vdots \\
b_i &= (0, \cdots, 0, -1, 0, \cdots, 0)^T = -e_i^T \\
&\vdots \\
b_n &= (0, \cdots, 0, -1, 0)^T = -e_n^T,
\end{aligned}
$$

where $\delta = \epsilon^{n+1}$. Since $det(L) = \delta = \epsilon^{n+1}$ and $dim(L) = n + 1$, by Minkowski's Corollary there exists $a \in L, a \neq 0$ s.t. $\|a\|_\infty \leq (det(L))^{\frac{1}{n+1}} = \epsilon$. Hence, there exist $q, p_1, \cdots, p_n \in \mathbb{Z}$ s.t. $a = qb_0 + \sum_{i=1}^n p_i b_i$ with $|a_i| \leq \epsilon$ for all i, or, equivalently,

1. $|a_i| = |q\alpha_i - p_i| \leq \epsilon$.
2. $a_n = q\delta \leq \epsilon$, or, equivalently, $q \leq \epsilon^{-n} \leq M$.

To complete the proof, we need only check that $q > 0$, (w.l.o.g. we can assume that $q \geq 0$ since we can always take $-a$ instead of a). Now, if $q = 0$ then by 1., $|p_i| \leq \epsilon$ for all i. But we know that $p_i \in \mathbb{Z}$ and that $p_i \neq 0$ for some i since $a \neq 0$. However, this contradicts the fact that $0 < \epsilon < 1$. □

Unfortunately, the proof is not constructive, since Minkowski's Corollary insures the existence of a, but doesn't give a procedure for finding it. However, if we make a stronger restriction on the value of M we can find a polynomial time solution to the problem.

5.1 Polynomial Time Algorithm for approximating SDA

We solve the following problem:

> Given $0 < \epsilon < 1, \alpha_1, \cdots, \alpha_n \in \mathbb{Q}$ find $p_1, \cdots, p_n, q \in \mathbb{Z}$ such that $0 < q \leq 2^{\frac{n(n+1)}{4}} \epsilon^{-n}$ and $|q\alpha_i - p_i| \leq \epsilon$ for all i.

This is a weaker version of the problem, but it can be solved in polynomial time.

To prove this we make use of the L^3 algorithm. But now we use $\delta = 2^{-\frac{n(n+1)}{4}} \epsilon^{n+1}$ in the basis L defined above. Using L^3 we can find $c \in L, c \neq 0$ (the first vector of the

Lovász-Reduced Basis) s.t.

$$\begin{aligned}
\|c\|_\infty &\leq \|c\|_2 \\
&\leq 2^{n/4}(det(L))^{1/(n+1)} \\
&= 2^{n/4}2^{-n/4}\epsilon \\
&= \epsilon
\end{aligned}$$

Hence we can find $q, p_1, \cdots, p_n$ s.t. $c = qb_0 + \sum_{i=1}^{n} p_i b_i, |c_i| \leq \epsilon_i$ or, equivalently,

1. $|q\alpha_i - p_i| \leq \epsilon$
2. $q\delta \leq \epsilon$ or $q \leq 2^{\frac{n(n+1)}{4}}\epsilon^{-n}$

by solving a simultaneous equation which is done in polynomial time. Note that even though the lattice L is not integral the L^3 algorithm works. Another approach may be to transform the lattice L into an integer lattice before using the L^3 algorithm.

5.2 Maintaining "Simple" Inequalities

We now show that the approximations obtained by this algorithm do in fact maintain "simple" inequalities. Suppose we have an input vector $x \in \mathbb{Q}^n, x = (\alpha_1, \cdots, \alpha_n)$ and we run this vector through the SDA algorithm described above, yielding $y = (\frac{p_1}{q}, \cdots, \frac{p_n}{q})$. Then the following theorem holds:

Theorem 8 *If $ax \leq b$ where $b \in \mathbb{Z}, a \in \mathbb{Z}^n$ and $\sum |a_i| < \frac{1}{\epsilon}$, then $ay \leq b$.*

Proof:

$$\begin{aligned}
b - ay &= (b - ax) + a(x - y) \\
&\geq a(x - y) \qquad \{\text{since } ax \leq b\} \\
&= \sum_i a_i(x_i - y_i) \\
&\geq -(\sum_i |a_i|)\|x - y\|_\infty \\
&> -\frac{1}{\epsilon}\frac{\epsilon}{q} \\
&= -\frac{1}{q}
\end{aligned}$$

But $b - ay$ is rational with denominator equal to q. Therefore, $b - ay \geq 0$. □

5.3 Repairing "Approximate" Inequalities

We saw that a "simple" inequality on x will also hold for its approximation y obtained by our algorithm for simultaneous Diophantine approximation. In fact, if a "simple" inequality "almost" holds for x, then the inequality holds for y, once passed through the SDA algorithm.