The Probabilistic Method

Idea: to show an object with certain properties exists

- generate a random object
- prove it has properties with nonzero probability
- often, "certain properties" means "good solution to our problem"

Last time

- set balancing
- expanders

The Probabilistic Method for Expectations

Outline

- goal to show exists object of given "value"
- give distribution with greater "expected value"
- deduce goal

Max-Cut:

- Define
- NP-complete
- Approximation algorithms
- factor 2
- "expected performance," so doesn't really fit our RP/ZPP framework

Wiring

Sometimes, it's hard to get hands on a good probability distribution of random answers.

- Problem formulation
  - $\sqrt{n} \times \sqrt{n}$ *gate array*
  - Manhattan wiring
  - boundaries between gates
  - fixed width boundary means limit on number of crossing wires
  - optimization vs. feasibility: minimize max crossing number

- focus on single-bend wiring. two choices for route.
- Generalizes if you know about multicommodity max-flow

- Linear Programs, integer linear programs
  - Black box
  - Good to know, since great solvers exist in practice
  - Solution techniques in other courses
  - LP is polytime, ILP is NP-hard
  - LP gives hints—rounding.

- IP formulation
  - x_{i0} means x_i starts horizontal, x_{i1} vertical
  - $T_{b0} = \{i \mid \text{net } i \text{ through } b \text{ if } x_{i0}\}$
  - T_{b1}
  - IP

$$\begin{aligned} \min \quad & w \\ x_{i0} + x_{i1} &= 1 \\ \sum_{i \in T_{b0}} x_{i0} + \sum_{i \in T_{b1}} x_{i1} &\leq w \end{aligned}$$

- Solution $\hat{x}_{i0}$, $\hat{x}_{i1}$, value $\hat{w}$.
- rounding is Poisson vars, mean $\hat{w}$.
- For $\delta < 1$ (good approx) $\Pr[\geq (1+\delta)\hat{w}] \leq e^{-\delta^2 \hat{w}/4}$
- need $2n$ boundaries, so aim for prob. bound $1/2n^2$.
- solve, $\delta = \sqrt{(4 \ln 2n^2)/\hat{w}}$.
- So absolute error $\sqrt{8\hat{w} \ln n}$
  - Good ($o(1)$-error) if $\hat{w} \gg 8 \ln n$
  - Bad ($O(\ln n)$ error) if $\hat{w} = 2$ (invoke other chernoff bound)
  - General rule: randomized rounding good if target logarithmic, not if constant

MAX SAT

Define.

- literals
- clauses
- NP-complete

random set

- achieve $1 - 2^{-k}$
- very nice for large k, but only 1/2 for $k = 1$

LP

$$\begin{aligned} \max & \quad \sum z_j \\ \sum_{i \in C_j^+} y_i + \sum_{i \in C_j^-} (1 - y_1) & \geq z_j \end{aligned}$$

Analysis

- $\beta_k = 1 - (1 - 1/k)^k$. values $1, 3/4, .704, \ldots$
- Random round y_i
- Lemma: k-literal clause sat w/pr at least $\beta_k \hat{z}_j$.
- proof:
  - assume all positive literals.
  - prob $1 - \prod(1 - y_i)$
  - maximize when all $y_i = \hat{z}_j / k$.
  - Show $1 - (1 - z/k)^k \geq \beta_k z$.
  - concave, so check equality at $z = 0, 1$
- Result: $(1 - 1/e)$ approximation (convergence of $(1 - 1/k)^k$)
- much better for small k: i.e. 1-approx for $k = 1$

LP good for small clauses, random for large.

- Better: try both methods.
- n_1, n_2 number in both methods
- Show $(n_1 + n_2)/2 \geq (3/4) \sum \hat{z}_j$
- $n_1 \geq \sum_{C_j \in S^k} (1 - 2^{-k}) \hat{z}_j$
- $n_2 \geq \sum \beta_k \hat{z}_j$
- $n_1 + n_2 \geq \sum (1 - 2^{-k} + \beta_k) \hat{z}_j \geq \sum \frac{3}{2} \hat{z}_j$

Method of Conditional Probabilities and Expectations

Derandomization.

- Theory: is P=RP?
- practice: avoid chance of error, chance of slow.

Conditional Expectation. Max-Cut

- Imagine placing one vertex at a time.
- $x_i = 0$ or 1 for left or right side
- $E[C] = (1/2)E[C|x_1 = 0] + (1/2)E[C|x_1 = 1]$
- Thus, either $E[C|x_1 = 0]$ or $E[C|X_1 = 1] \geq E[C]$
- Pick that one, continue
- More general, whole tree of element settings.
  - Let $C(a) = E[C \mid a]$.
  - For node a with children b, c, either $C(b)$ or $C(c) \geq C(a)$.
- By induction, get to leaf with expected value at least $E[C]$
- But no randomness left, so that is actual cut value.
- Problem: how compute node values? Easy.

Conditional Probabilities. Set balancing. (works for wires too)

- Review set-balancing Chernoff bound
- Think of setting item at a time
- Let Q be bad event (unbalanced set)
- We know $\Pr[Q] < 1/n$.
- $\Pr[Q] = 1/2 \Pr[Q \mid x_{i0}] + 1/2 \Pr[Q \mid x_{i1}]$
- Follows that one of conditional probs. less than $\Pr[Q] < 1/n$.
- More general, whole tree of element settings.
  - Let $P(a) = \Pr[Q \mid a]$.
  - For node a with children b, c, $P(b)$ or $P(c) < P(a)$.
  - $P(r) < 1$ sufficient at root r.
  - at leaf l, $P(l) = 0$ or 1.
- One big problem: need to compute these probabilities!

Pessimistic Estimators.

- Alternative to computing probabilities
- three neceessary conditions:
  - $\hat{P}(r) < 1$
  - $\min\{\hat{P}(b), \hat{P}(c)\} < \hat{P}(a)$
  - $\hat{P}$ computable

  Imply can use $\hat{P}$ instead of actual.
- Let $Q_i = \Pr[\text{unbalanced set } i]$
- Let $\hat{P}(a) = \sum \Pr[Q_b \mid a]$ at tree node a
- Claim 3 conditions.
  - HW
- Result: deterministic $O(\sqrt{n \ln n})$ bias.
- more sophisticated pessimistic estimator for wiring.

Oblivious routing

- recall: choose random routing. Only $1/N$ chance of failure
- Choose N^3 random routines.
- whp, for every permutation, at most $2N^2$ bad routes.
- given the N^3 routes, pick one at random.
- so for any permutation, prob $2/N$ of being bad.
- Advantage: N^3 routes can be stored