MIT 6.972 Algebraic techniques and semidefinite optimization May 2, 2006

Lecture 19

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Today we continue with some additional aspects of quantifier elimination. We will then recall the Positivstellensatz and its relations with semidefinite programming. After introducing copositive matrices, we present Pólya's theorem on positive forms on the simplex, and the associated relaxations. Finally, we conclude with an important result due to Schmüdgen about representation of positive polynomials on compact sets.

1 Certificates

`Talk about certificates in QE` ToDo

2 Psatz revisited

Recall the statement of the Positivstellensatz.

Theorem 1 (Positivstellensatz). *Consider the set* $S = \{x \in \mathbb{R}^n \mid f_i(x) \geq 0,\ h_i(x) = 0\}$. *Then,*

$$S = \emptyset \qquad \Leftrightarrow \qquad \exists f, h \in \mathbb{R}[x] \ s.t. \begin{cases} f + h & = & -1 \\ f & \in & \mathbf{cone}\{f_1, \ldots, f_s\} \\ h & \in & \mathbf{ideal}\{h_1, \ldots, h_t\} \end{cases}$$

Once again, since the conditions on the polynomials f, h are convex and affine, respectively, by restricting their degree to be less than or equal to a given bound d we have a finite-dimensional semidefinite programming problem.

2.1 Hilbert 17th problem

As we have seen, in the general case nonnegative multivariate polynomials can fail to be a sum of squares (the Motzkin polynomial being the classical counterxample). As part of his famous list of twenty-three problems that he presented at the International Congress of Mathematicians in 1900, David Hilbert asked the following[1]:

> **17. Expression of definite forms by squares.** A rational integral function or form in any number of variables with real coefficient such that it becomes negative for no real values of these variables, is said to be definite. The system of all definite forms is invariant with respect to the operations of addition and multiplication, but the quotient of two definite forms in case it should be an integral function of the variables is also a definite form. The square of any form is evidently always a definite form. But since, as I have shown, not every definite form can be compounded by addition from squares of forms, the question arises which I have answered affirmatively for ternary forms whether every definite form may not be expressed as a quotient of sums of squares of forms. At the same time it is desirable, for certain questions as to the possibility of certain geometrical constructions, to know whether the coefficients of the forms to be used in the expression may always be taken from the realm of rationality given by the coefficients of the form represented.

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[1] This text was obtained from `http://aleph0.clarku.edu/~djoyce/hilbert/`, and corresponds to Newson's translation of Hilbert's original German address. In that website you will also find links to the current status of the problems, as well as the original German text.

In other words, can we write every nonnegative polynomial as a sum of squares of *rational functions*? As we we show next, this is a rather direct consequence of the Psatz. Of course, it should be clear (and goes without saying) that we are (badly) inverting the historical order! In fact, much of the motivation for the development of real algebra came from Hilbert's question.

How can we use the Psatz to prove that a polynomial $p(x)$ is nonnegative? Clearly, p is nonnegative if and only if the set $\{x \in \mathbb{R}^n \,|\, p(x) < 0\}$ is empty. Since our version of the Psatz does not allow for strict inequalities (there are slightly more general, though equivalent, formulations that do), we'll need a useful trick discussed earlier ("Rabinowitch's trick"). Introducing a new variable z, the nonnegativity of $p(x)$ is equivalent to the emptiness of the set described by

$$-p(x) \geq 0, \qquad 1 - zp(x) = 0.$$

The Psatz can be used to show that this holds if and only if there exist polynomials $s_0, s_1, t \in \mathbb{R}[x, z]$ such that

$$s_0(x, z) - s_1(x, z) \cdot p + t(x, z) \cdot (1 - zp) = -1,$$

where s_0, s_1 are sums of squares. Replace now $z \to 1/p(x)$, and multiply by p^{2k} (where k is sufficiently large) to obtain

$$\tilde{s}_0 - \tilde{s}_1 \cdot p = -p^{2k},$$

where $\tilde{s}_0, \tilde{s}_1$ are sums of squares in $\mathbb{R}[x]$. Solving now for p, we have:

$$p(x) = \frac{\tilde{s}_0(x) + p(x)^{2k}}{\tilde{s}_1(x)} = \frac{\tilde{s}_1(x)(\tilde{s}_0(x) + p(x)^{2k})}{\tilde{s}_1^2(x)},$$

and since the numerator is a sum of squares, it follows that $p(x)$ is indeed a sum of squares of rational functions.

3 Copositive matrices and Pólya's theorem

An interesting class of matrices are the so-called *copositive matrices*, which are those for which the associated quadratic form is nonnegative on the nonnegative orthant.

Definition 2. *A matrix $M \in \mathcal{S}^n$ is* copositive *is it satisfies*

$$x^T M x \geq 0, \qquad \textit{for all} \quad x_i \geq 0.$$

As opposed to positive semidefiniteness, which can be checked in polynomial time, the recognition problem for copositive matrices is an NP-hard problem. The set of copositive is a closed convex cone, for which checking membership is a difficult problem.

There are many interesting applications of copositive matrices. Among others, we mention:

- Consider a graph G, with A being its the adjacency matrix. The stability number α of the graph G is equal to the cardinality of its largest stable set. By a result of Motzkin and Straus, it is known that it can be obtained as:

  $$\frac{1}{\alpha(G)} = \min_{x_i \geq 0, \sum_i x_i = 1} x^T (I + A) x$$

  This implies that $\alpha(G) \leq \gamma$ if and only if the matrix $\gamma \cdot (I + A) - ee^T$ is copositive.

- Another interesting application of copositive matrices is in the performance analysis of queueing networks; see e.g. [KM96]. Modulo some (important) details, the basic idea is to use a quadratic function $x^T M x$ as a Lyapunov function, where the matrix M is copositive and x represents the lengths of the queues.

`Add details + others` ToDo

An important related result is Pólya's theorem on positive forms on the simplex:

Theorem 3 (Pólya)**.** *Consider a homogeneous polynomial in n variables of degree d, that is strictly positive in the unit simplex $\Delta_n := \{x \in \mathbb{R}^n \,|\, x_i \geq 0, \sum_{i=1}^n x_i = 1\}$. Then, for large enough k, the polynomial $(x_1 + \cdots + x_n)^k p(x)$ has nonnegative coefficients.*

We can provide a natural hierarchy of sufficient conditions for a matrix to be copositive. Completeness of this hierarchy follows directly from Pólya's theorem [Par00].

There are some very interesting connections between Pólya's result and a foundational theorem in probability known as De Finetti's exchangeability theorem.

4 Positive polynomials

The Positivstellensatz allows us to obtain certificates of the emptiness of a basic semialgebraic set, explicitly given by polynomials.

What if we want to apply this for optimization? As we have seen, it is relatively straightforward to convert an optimization problem to a family of feasibility problems, by considering the sublevel sets, i.e., the sets $\{x \in \mathbb{R}^n \,|\, f(x) \leq \gamma\}$.

In the general case of constrained problems, however, using the Psatz we will require conditions that are not linear in the unknown parameter γ (because we need products between the contraints), and this presents a difficulty to the direct use of SDP. Notice nevertheless, that the problem is certainly an SDP for any fixed value of γ, and it thus quasiconvex (which is almost as good, except for the fact that we cannot use "standard" SDP solvers to solve it directly, but rather rely on methods such as bisection).

Theorem 4 ([Sch91])**.** *If $p(x)$ is strictly positive on $K = \{x \in \mathbb{R}^n \,|\, f_i(x) \geq 0\}$, and K is compact, then $p(x) \in \mathbf{cone}\{f_1, \ldots, f_s\}$.*

In the next lecture we will describe the basic elements of Schmüdgen's proof. His approach combines both algebraic tools (using the Positivstellensatz to prove the boundedness of certain operators) and functional analysis (spectral measures of commuting families of operators and the Hahn-Banach theorem). We will also describe some alternative versions due to Putinar, as well as a related purely functional-analytic result due to Megretski.

For a comprehensive treatment and additional references, we mention [BCR98, Mar00, PD01] among others.

References

[BCR98] J. Bochnak, M. Coste, and M-F. Roy. *Real Algebraic Geometry.* Springer, 1998.

[KM96] P. R. Kumar and S. P. Meyn. Duality and linear programs for stability and performance analysis of queuing networks and scheduling policies. *IEEE Trans. Automat. Control*, 41(1):4–17, 1996.

[Mar00] M. Marshall. *Positive polynomials and sums of squares.* Dottorato de Ricerca in Matematica. Dept. di Mat., Univ. Pisa, 2000.

[Par00] P. A. Parrilo. *Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization.* PhD thesis, California Institute of Technology, May 2000. Available at `http://resolver.caltech.edu/CaltechETD:etd-05062004-055516`.

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