ESD.86

Markov Processes and their Application to Queueing

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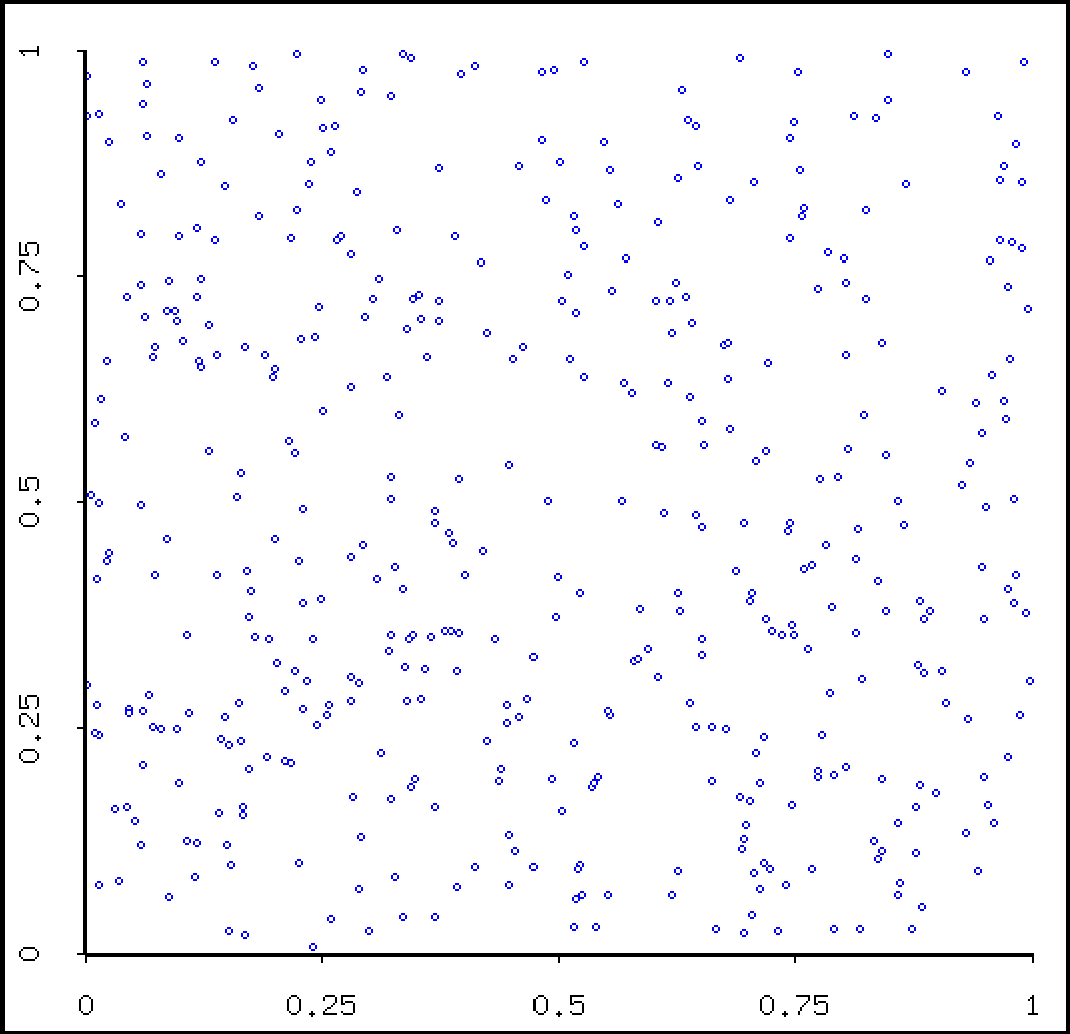
Outline

- Spatial Poisson Processes, one more time
- Introduction to Queueing Systems
- Little's Law
- Markov Processes

Spatial Poisson Processes



Courtesy of Andy Long. Used with permission.

http://zappa.nku.edu/~longa/geomed/modules/ss1/lec/poisson.gif

Spatial Poisson Processes

- Entities distributed in space (Examples?)
- Follow postulates of the (time) Poisson process
  - *λdt = Probability of a Poisson event in dt*
  - *History not relevant*
  - *What happens in disjoint time intervals is independent, one from the other*
  - *The probability of a two or more Possion events in dt is second order in dt and can be ignored*

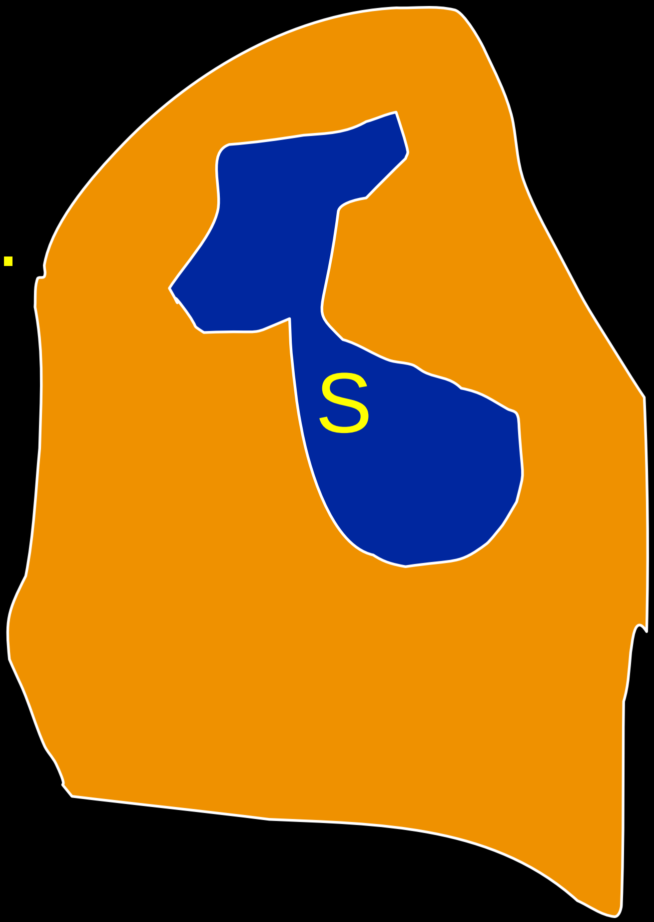
- Let's fill in the spatial analogue…..

Set *S* has area *A*(*S*).

Poisson intensity is γ Poisson entities/(unit area).

X(*S*) is a random variable

X(*S*) = number of Poisson entities in *S*



$$P\{X(S)=k\}=\frac{(\gamma A(S))^k}{k!}e^{-\gamma A(S)},\ k=0,1,2,\ldots$$

Nearest Neighbors: Euclidean

Define D_2= distance from a random point to nearest Poisson entity

Want to derive $f_{D_2}(r)$.

Happiness:

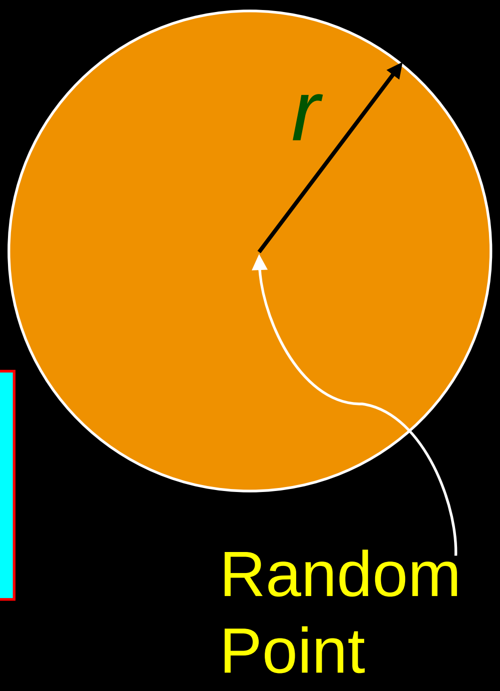
$$F_{D_2}(r) \equiv P\{D_2 \le r\} = 1 - P\{D_2 > r\}$$

$$F_{D_2}(r) = 1 - \Pr ob\{\text{no Poisson entities in circle of radius } r\}$$

$$F_{D_2}(r) = 1 - e^{-\gamma \pi r^2} \quad r \ge 0$$

$$f_{D_2}(r) = \frac{d}{dr} F_{D_2}(r) = 2r\gamma\pi e^{-\gamma\pi r^2} \quad r \ge 0$$

Rayleigh pdf with parameter $\sqrt{2\gamma\pi}$



Nearest Neighbors: Euclidean

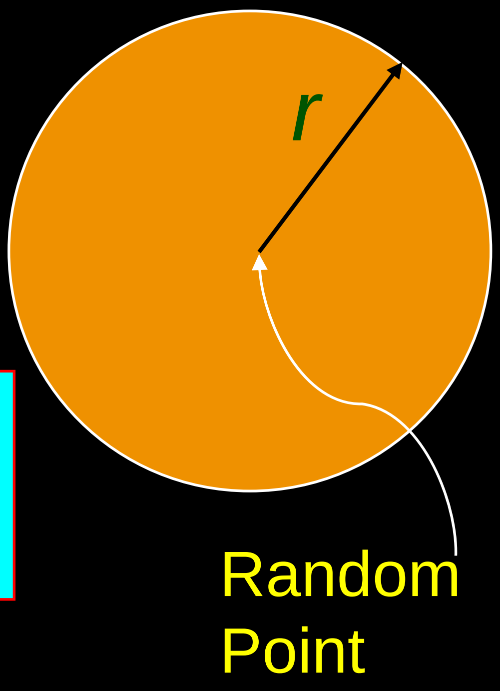
Define D_2 = distance from a random point to nearest Poisson entity

Want to derive $f_{D_2}(r)$.

$$E[D_2] = (1/2)\sqrt{\frac{1}{\gamma}} \quad \text{"Square Root Law"}$$

$$\sigma^2_{D_2} = (2 - \pi/2)\frac{1}{2\pi\gamma}$$

$$f_{D_2}(r) = \frac{d}{dr}F_{D_2}(r) = 2r\gamma\pi e^{-\gamma\pi r^2} \quad r \geq 0$$

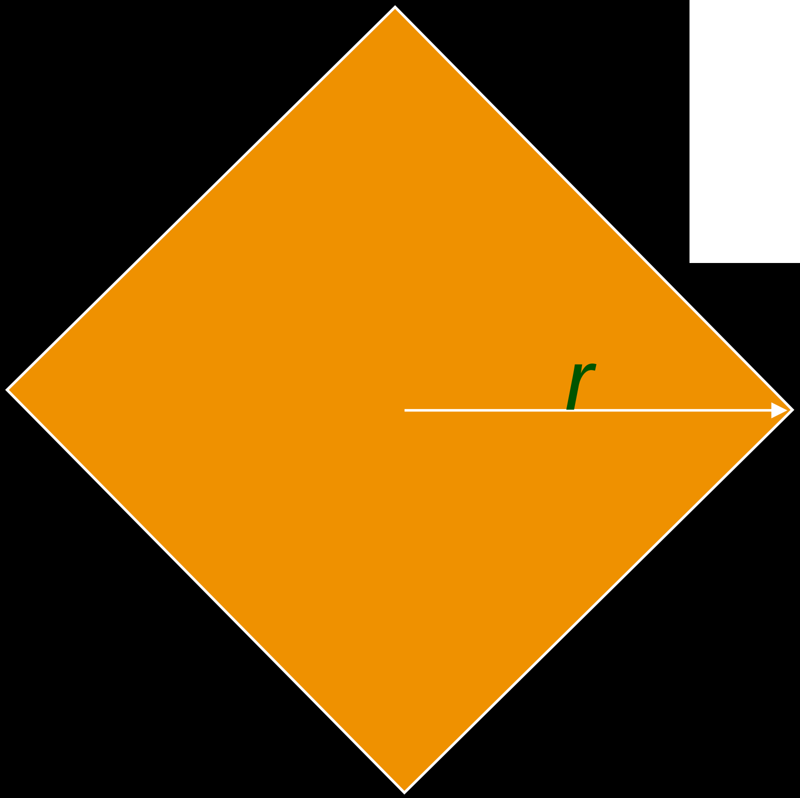


Rayleigh pdf with parameter $\sqrt{2\gamma\pi}$

Nearest Neighbor: Taxi Metric

$$F_{D_1}(r) \equiv P\{D_1 \leq r\}$$

$$F_{D_1}(r) = 1 - \Pr\{\text{no Poisson entities in diamond}\}$$



How Might you Derive the PDF fo the k^{th} Nearest Neighbor?

Blackboard exercise!

To Queue or Not to Queue, That May be a Question!

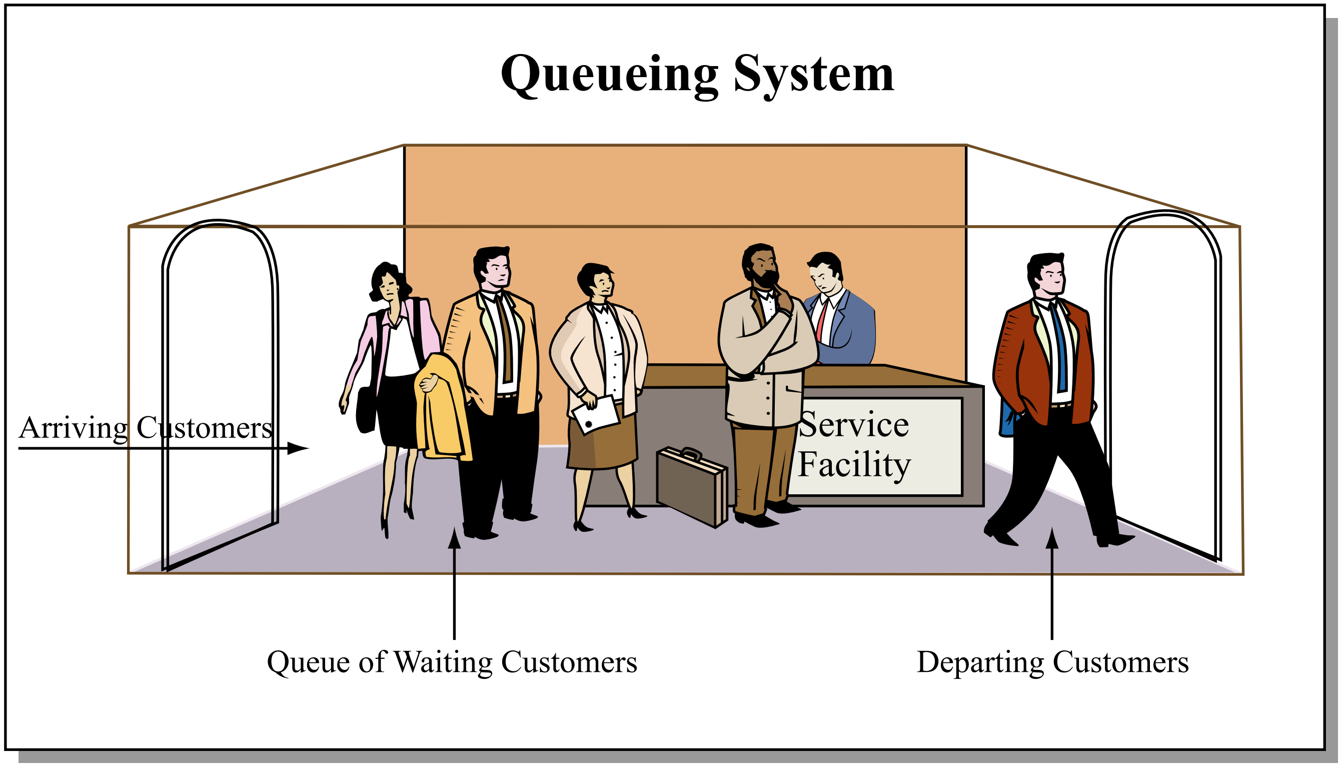
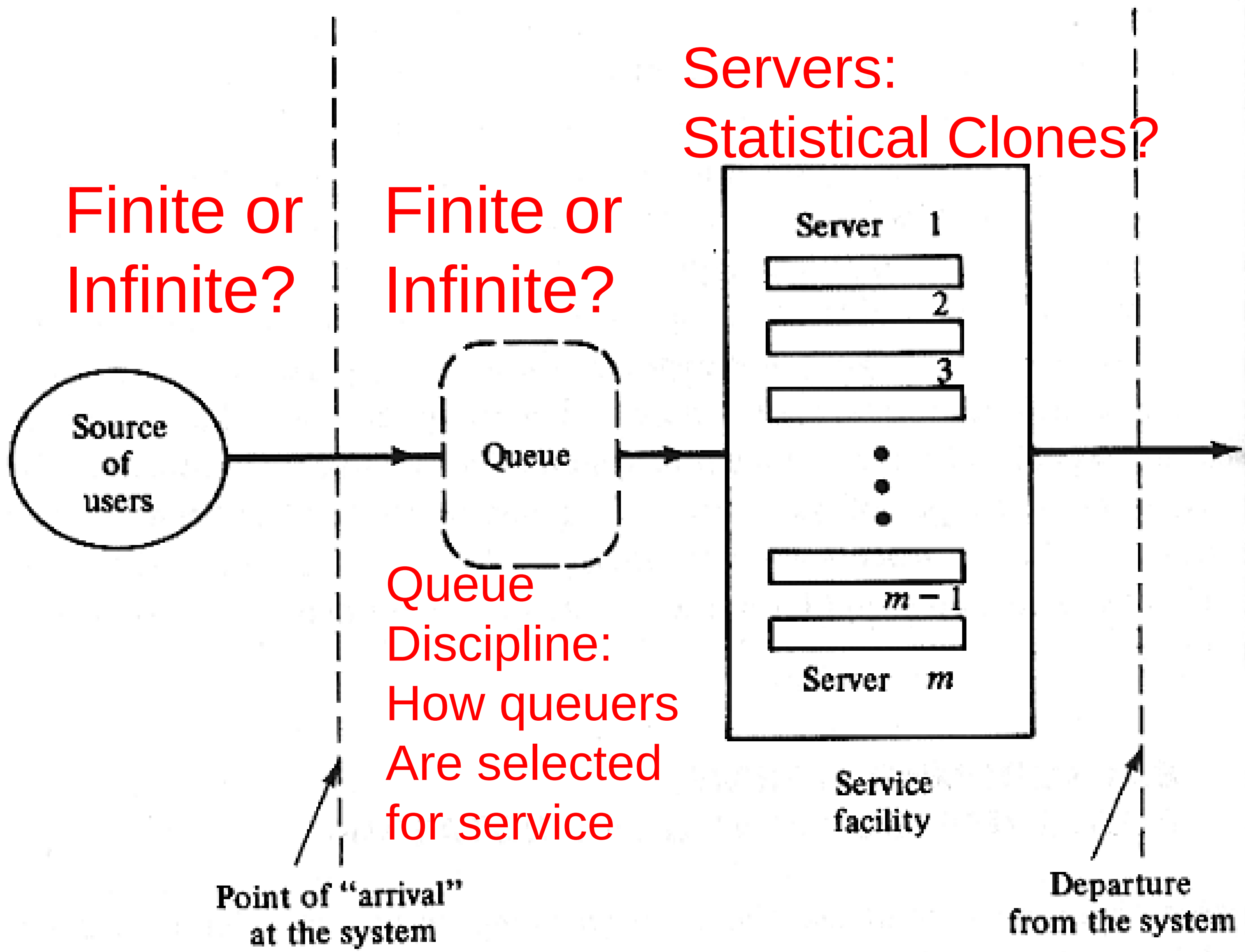


Figure by MIT OCW.



Generic queueing system.

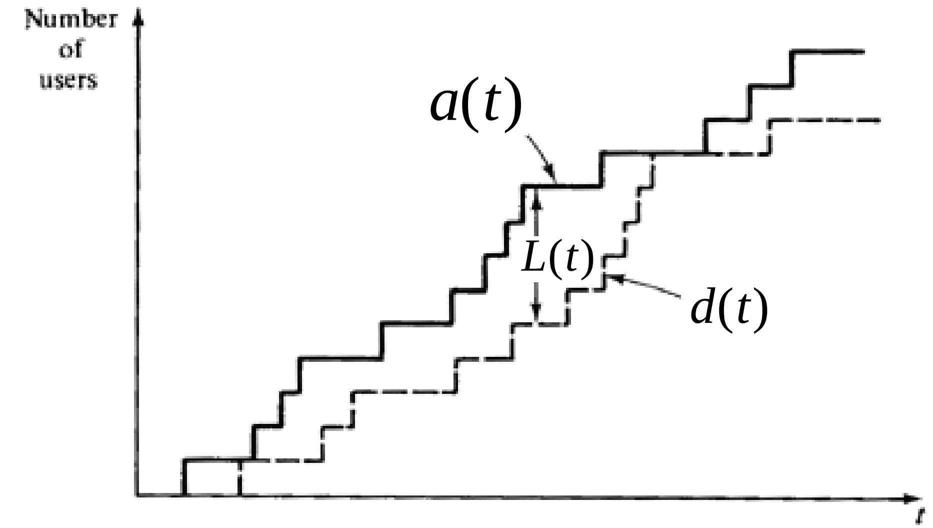
Source: Larson and Odoni, *Urban Operations Research*

What Kinds of Queues Occur in Systems of Interest to ESD?

ESD Queues?

Photos courtesy, from top left, clockwise: U.S. FAA: Flickr user "*keng" http://www.flickr.com/photos/kengz/67187556/; Luke Hoersten http://www.flickr.com/photos/lukehoersten/532375235/)

Little's Law for Queues



Arrivals and service completions at a queueing system.

Source: Larson and Odoni, *Urban Operations Research*

$a(t)$ = cumulative # arrivals to system in $(0,t]$

$d(t)$ = cumulative # departures from system in $(0,t]$

$L(t) = a(t) - d(t)$

$L(t)$ = number of customers in the system (in queue and in service) at time t

Little's Law for Queues

$$\gamma(t) = \int_0^t [a(\tau) - d(\tau)]d\tau = \int_0^t L(\tau)d\tau$$

$$\gamma(t) = \text{total number of customer minutes spent in the system}$$

$$a(t) = \text{cumulative \# arrivals to system in } (0,t]$$

$$d(t) = \text{cumulative \# departures from system in } (0,t]$$

$$L(t) = a(t) - d(t)$$

$$L(t) = \text{number of customers in the system}$$

$$\text{(in queue and in service) at time } t$$

Let’s Get an expression for Each of 3 Quantities

$\lambda_t \equiv \text{average customer arrival rate} = a(t)/t$

$$L_t = \frac{\gamma(t)}{t} = \frac{a(t)}{t}\frac{\gamma(t)}{a(t)} = \lambda_t W_t$$

In the limit,

$L = \lambda W$,   Little's Law

Key Issues

$$L = \lambda W$$

- *L* in a time-average. Explain
- λ is average of arrival rate of customers who actually enter the system
- *W* is average time in system (in queue and in service) for actual customers who enter the system

More Issues

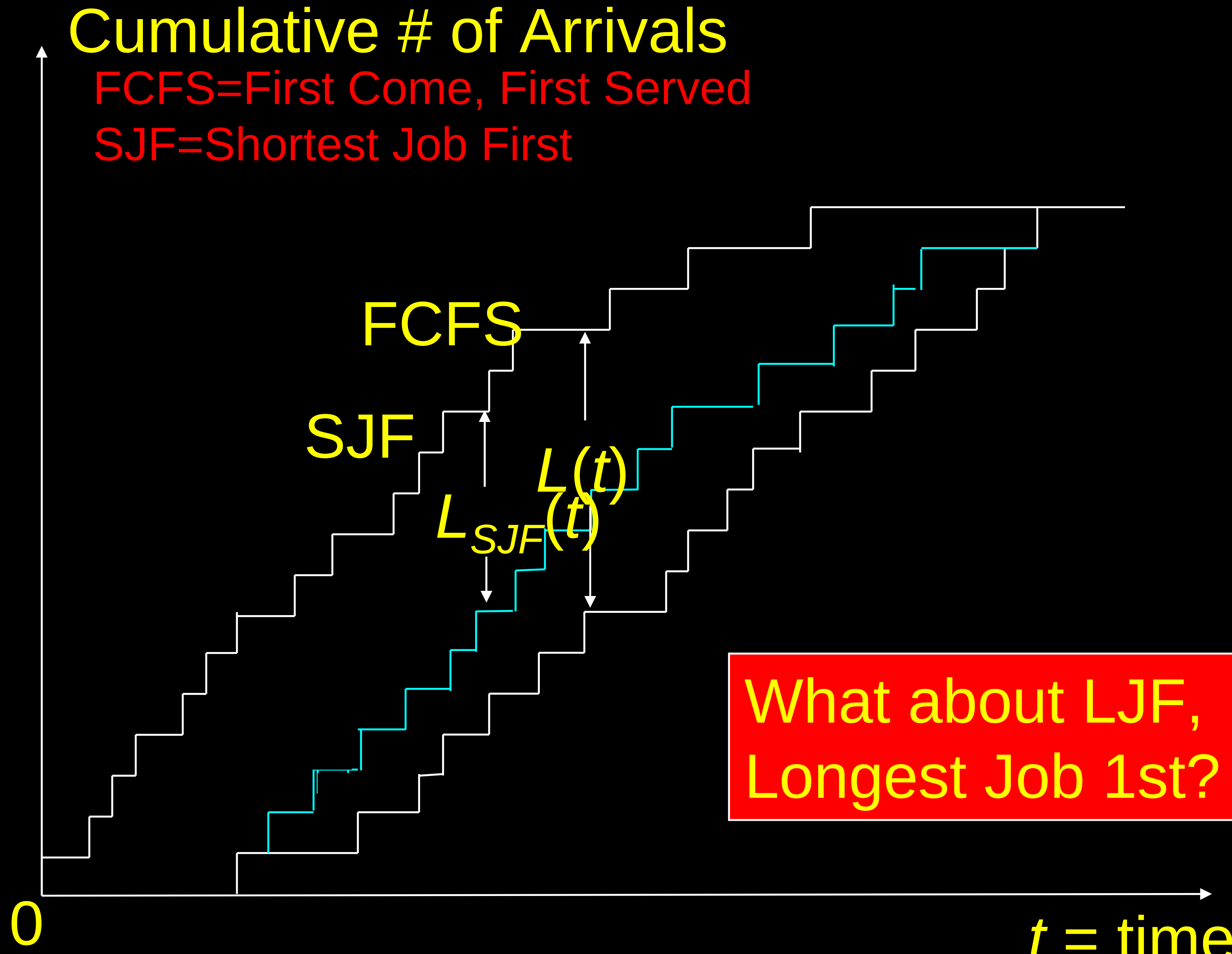
$$L = \lambda W$$

- *Little's Law is general. It does not depend on*
  - – Arrival process
  - – Service process
  - – # servers
  - – Queue discipline
  - – Renewal assumptions, etc.
- *It just requires that the 3 limits exist.*

Still More Issues

$$L = \lambda W$$

- *What about balking? Reneging? Finite capacity?*
- *Do we need iid service times? Iid inter-arrival times?*
- *Do we need each busy period to behave statistically identically?*
- *Look at role of $\gamma(t)$. Can change queue statistics by changing queue discipline.*

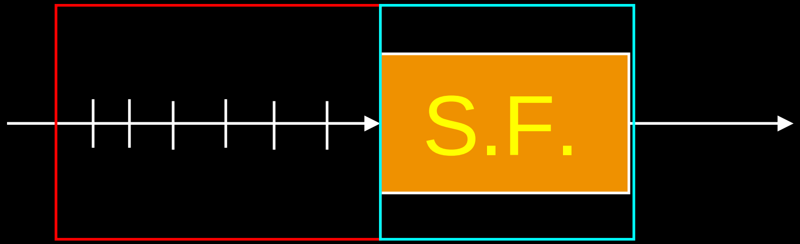


What about LJF, Longest Job 1st?

"System" is General

$$L = \lambda W$$

- *Our results apply to entire queue system, queue plus service facility*
- *But they could apply to queue only!*



$$L_q = \lambda W_q$$

- *Or to service facility only!*

$$L_{SF} = \lambda W_{SF} = \lambda / \mu$$

$$1/\mu = \text{mean service time}$$

All of this means,
"You buy one, you get the other 3 for free!"

$$W = \frac{1}{\mu} + W_q$$

$$L = L_q + L_{SF} = L_q + \frac{\lambda}{\mu}$$

$$L = \lambda W$$

Utilization Factor ρ

- Single Server. Set $Y = \begin{cases} 1 \text{ if server is busy} \\ 0 \text{ if server is idle} \end{cases}$

$$E[Y] = 1 * P\{\text{server is busy}\} + 0 * P\{\text{server is idle}\}$$

$$E[Y] = 1 * \rho + 0 = \rho = E[\#\ \text{customers in SF}] = ?$$

- E[Y] is time-average number of customers in the SF
- Buy Little's Law,

$$\rho = \lambda / \mu < 1$$

Utilization Factor ρ

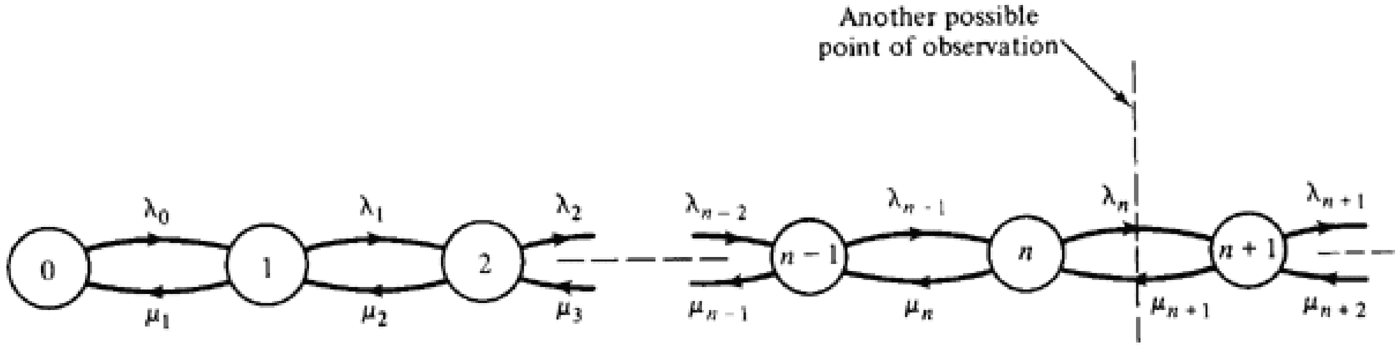
- Similar logic for *N* identical parallel servers gives

$$\rho = \left(\frac{\lambda}{N}\right)\frac{1}{\mu} = \frac{\lambda}{N\mu} < 1$$

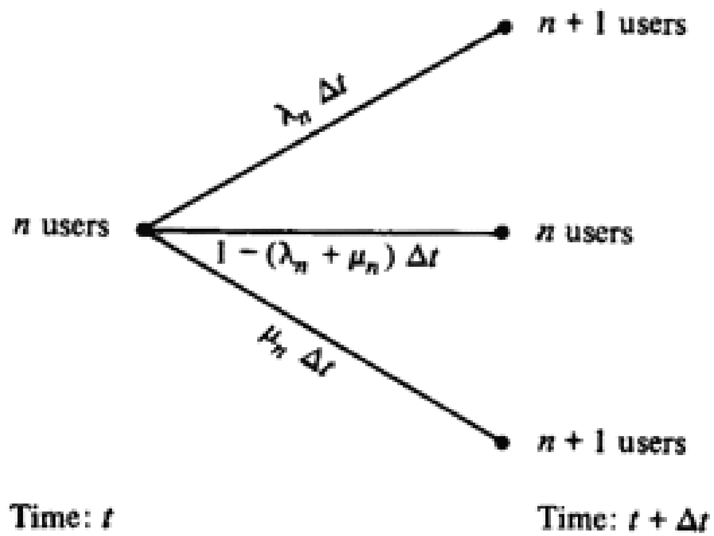
- Here, λ/μ corresponds to the time-average number of servers busy

Markov Queues

Markov here means, “No Memory”



State-transition diagram for the fundamental birth-and-death model.



Probabilities of transitions for birth-and-death model in time Δt.

Source: Larson and Odoni, *Urban Operations Research*

Balance of Flow Equations

$$\lambda_0 P_0 = \mu_1 P_1$$

$$(\lambda_n + \mu_n) P_n = \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1} \text{ for } n = 1, 2, 3, \ldots$$

To be continued…………..