

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Departments of Electrical Engineering, Mechanical Engineering, and the Harvard-MIT Division
of Health Sciences and Technology

6.022J/2.792J/BEH.371J/HST542J: Quantitative Physiology: Organ Transport Systems

PROBLEM SET 1

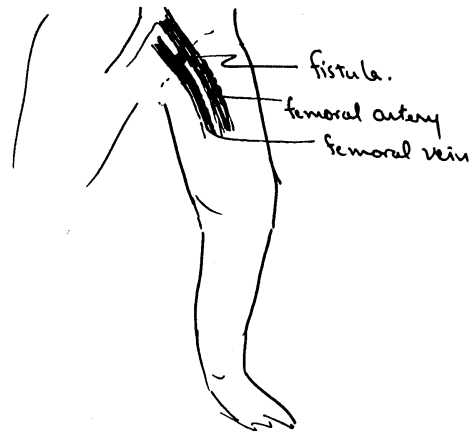
SOLUTIONS

February 10, 2004

Problem 1

Certain traumatic injuries such as gunshots or knife wounds to the groin may lead to the development of a “short circuit” or fistula between the large femoral artery and the adjacent femoral vein. (See Figure 1.) The fistula may have a very low resistance to fluid flow, and may carry large blood flow rates.

Figure 1:



Two medical students were arguing about the expected effect of a large femoral AV fistula on the blood supply to the leg. One student was certain that the leg would be deprived of its blood supply and would become gangrenous. The other student violently disagreed, and insisted that the leg would be in no danger. Who was correct?

- A. Propose a simple resistive model to represent the blood supply to the leg with and without the presence of the fistula. The model should include the aorta, the femoral artery, the leg circulation, and the corresponding veins. Assume that the resistance of the fistula (when open) is much lower than the resistance of the leg vascular bed, but higher than the (negligible) resistance of the aorta. Assume negligible resistance for the large veins as well as the aorta. Assume the aortic pressure, P_0 , is constant. The following variables may be used to model the system:

- R_A : Large Artery Resistance
- R_L : Leg Resistance
- R_F : Fistula Resistance
(venous resistance is negligible)
- P_0 : Constant Pressure Source
(due to baroreceptor reflexes)
- P_L : Pressure above Leg
Resistance
and Fistula Resistance
- Q_T : Cardiac Output (Total Flow)
- Q_F : Fistula Flow
- Q_L : Leg Flow

- B. Calculate the blood flow to the leg with and without the fistula, and referee the argument. Express your answers in terms of resistances and aortic pressure. Determine the ratio:

$$\frac{(\text{Blood flow to the leg with the fistula})}{(\text{Blood flow to the leg without the fistula})}$$

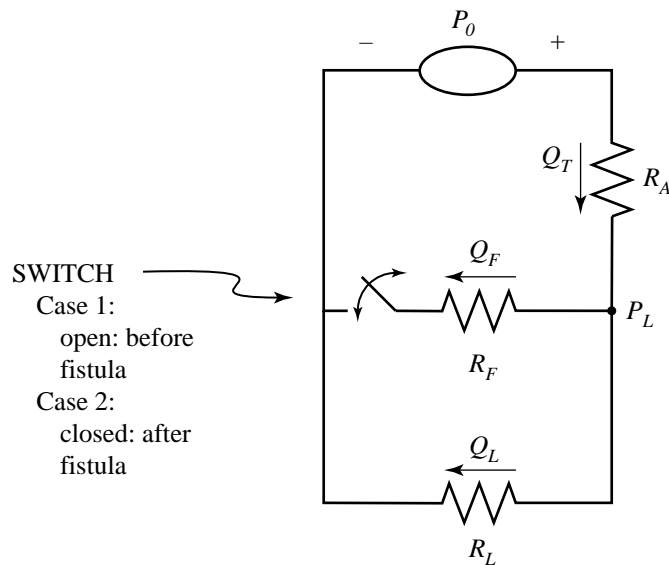
and plot this ratio as a function of relevant resistances, $\frac{R_A}{R_F}$.

- C. What will be the change in cardiac output when the fistula opens? Express this change as a ratio:

$$\frac{(\text{cardiac output with the fistula})}{(\text{cardiac output without the fistula})}$$

in terms of relevant resistances, $\frac{R_A}{R_F}$.

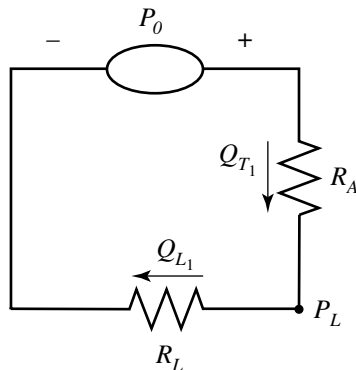
Resistive Network



- R_A : Large Artery Resistance
- R_L : Leg Resistance
- R_F : Fistula Resistance
(venous resistance is negligible)
- P_0 : Constant Pressure Source
(due to baroreceptor reflexes)
- P_L : Pressure above Leg
Resistance and Fistula Resistance
- Q_T : Cardiac Output (Total Flow)
- Q_F : Fistula Flow
- Q_L : Leg Flow

Case 1

Before AV fistula: Switch open:



$$Q_{T1} = Q_{L1}$$

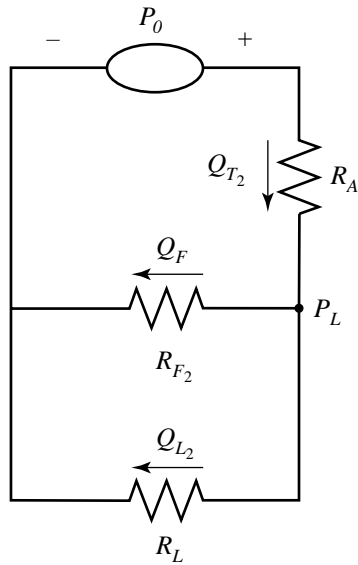
$$Q_{T1} = \frac{P_0}{R_A + R_L} = Q_{L1}$$

If $R_L \gg R_A$, then

$$Q_{T1} = Q_{L1} = \frac{P_0}{R_L}$$

Case 2

After AV fistula: Switch closed:



$$P_L = P_0 - Q_{T_2} R_A$$

$$Q_{T_2} = Q_{F_2} + Q_{L_2} = \frac{P_L}{R_F} + \frac{P_L}{R_L}$$

Substitution:

$$Q_{T_2} = \frac{P_0}{R_A} + \frac{R_F R_L}{R_F + R_L}$$

$$Q_{L_2} = \frac{P_L}{R_L}$$

$$= Q_{T_2} \left(\frac{R_F}{R_F + R_L} \right)$$

$$= P_0 \left(\frac{R_F}{R_A(R_F + R_L) + R_F R_L} \right)$$

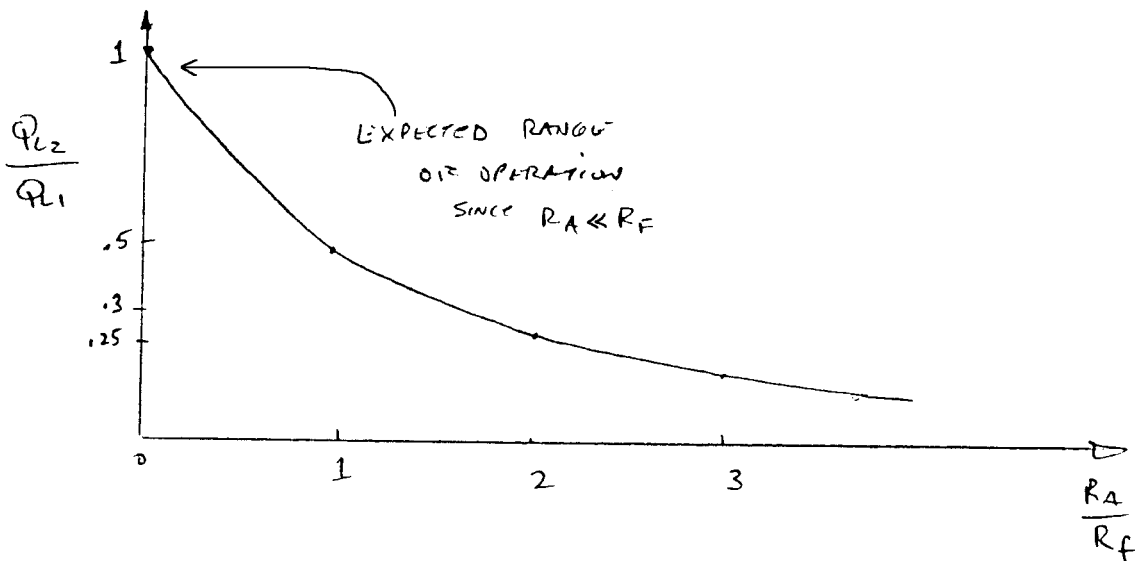
Comparing \$Q_{L_2}\$ and \$Q_{L_1}\$:

If \$R_L \gg R_A\$:

$$Q_{L_2} = \frac{P_0}{R_L \left(1 + \frac{R_A}{R_F} \right)} \quad Q_{L_1} = \frac{P_0}{R_L}$$

Flow of blood through the leg following fistula relative to the flow before the fistula:

$$\frac{Q_{L_2}}{Q_{L_1}} = \frac{1}{1 + \frac{R_A}{R_F}}$$



Thus:

If $R_A \ll R_F$, then leg flow is not affected by the shunt (the expected situation).

If $R_A \approx R_F$, then leg flow drops to 1/2 the control value.

Comparing the cardiac outputs:

If $R_L \gg R_A$:

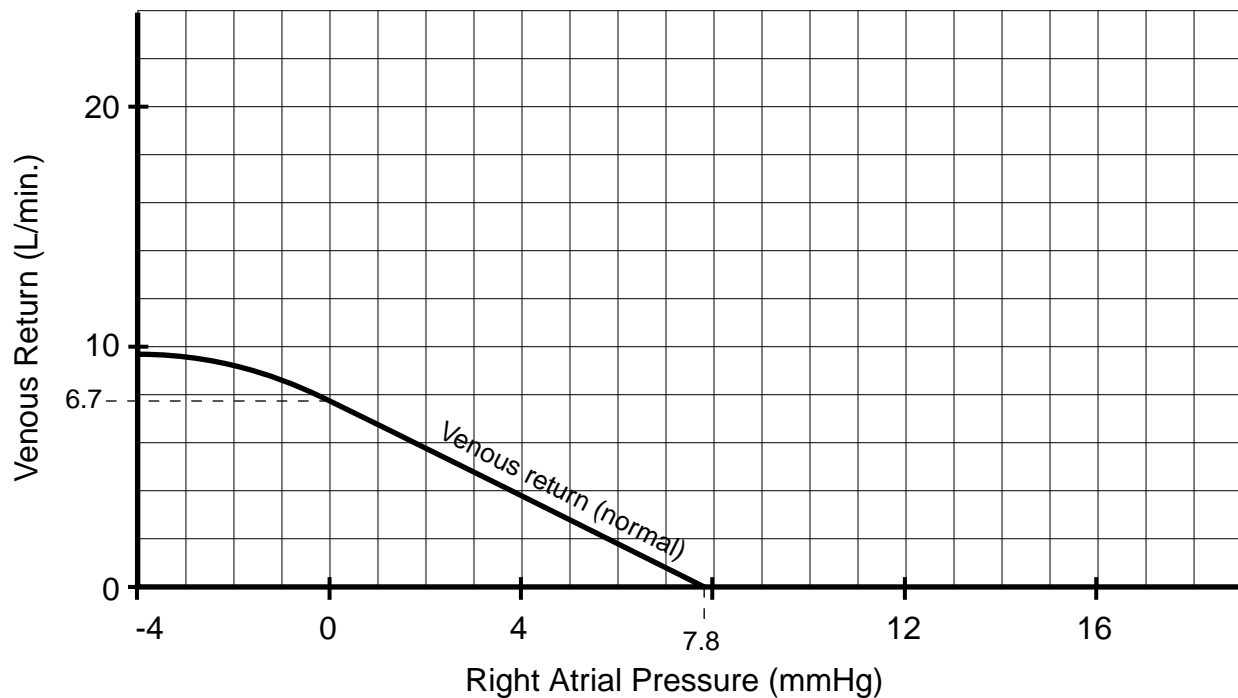
$$\begin{aligned}Q_{T_1} &\approx \frac{P_0}{R_L} \\Q_{T_2} &= \frac{P_0}{R_A + \frac{R_F R_L}{R_F + R_L}} \\ \frac{Q_{T_2}}{Q_{T_1}} &= \frac{1}{\frac{R_A}{R_L} + \frac{R_F}{R_F + R_L}} \approx \frac{R_F + R_L}{R_F} \\ \frac{Q_{T_2}}{Q_{T_1}} &\approx 1 + \frac{R_L}{R_F}\end{aligned}$$

So a fistula is manifested in the body by a huge increase in cardiac output, but little change in leg blood flow.

Problem 2

We have used the venous return curve to represent the function of the peripheral circulation. In this question you are asked to indicate the *quantitative* changes in each of the curves (such as change in slope or x -intercept.) Also, sketch your answers directly on the normal curves supplied for your reference. Please refer to Table 6 in the course notes, Cardiovascular Mechanics section, p. 84, for necessary parameter values. Assume that no extrinsic control mechanisms are operating.

A. Heart rate increases from 67 bpm to 100 bpm.

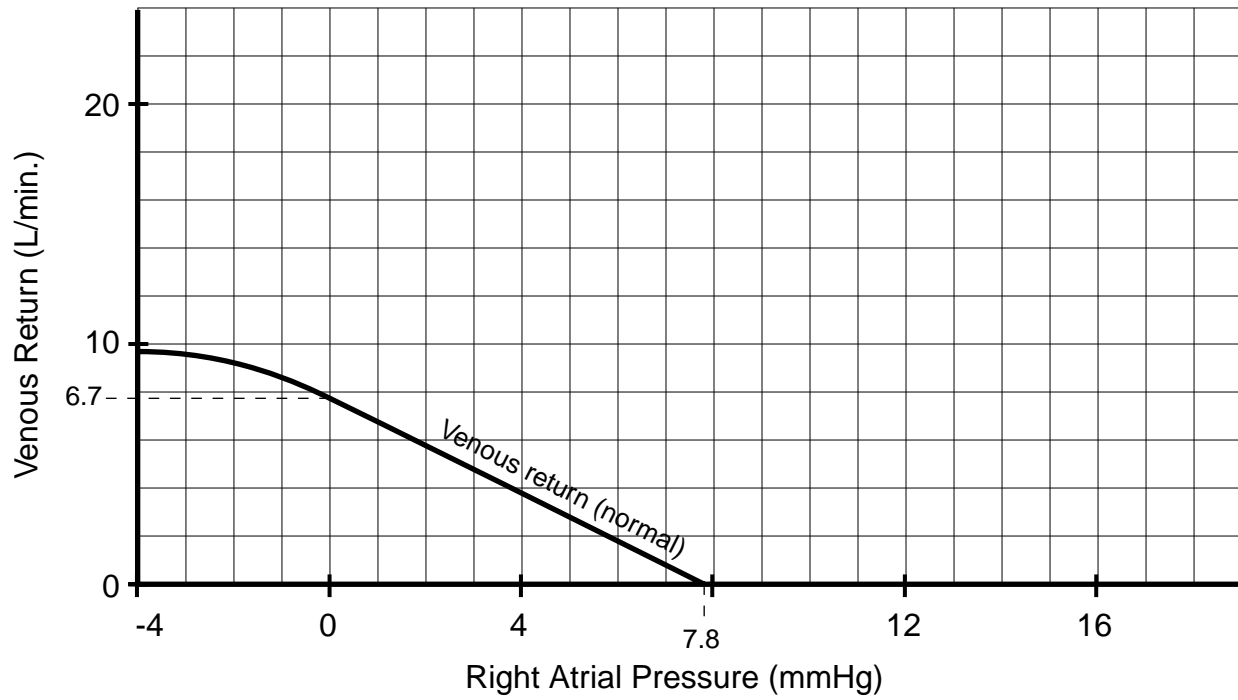


No change in venous return curve. Same as original.

$$Q = \frac{P_{MS} - P_f}{R_v + R_a \left(\frac{C_a}{C_a + C_v} \right)}$$

B. The systemic venous zero-pressure volume decreases to 2500 mL (with no change in total

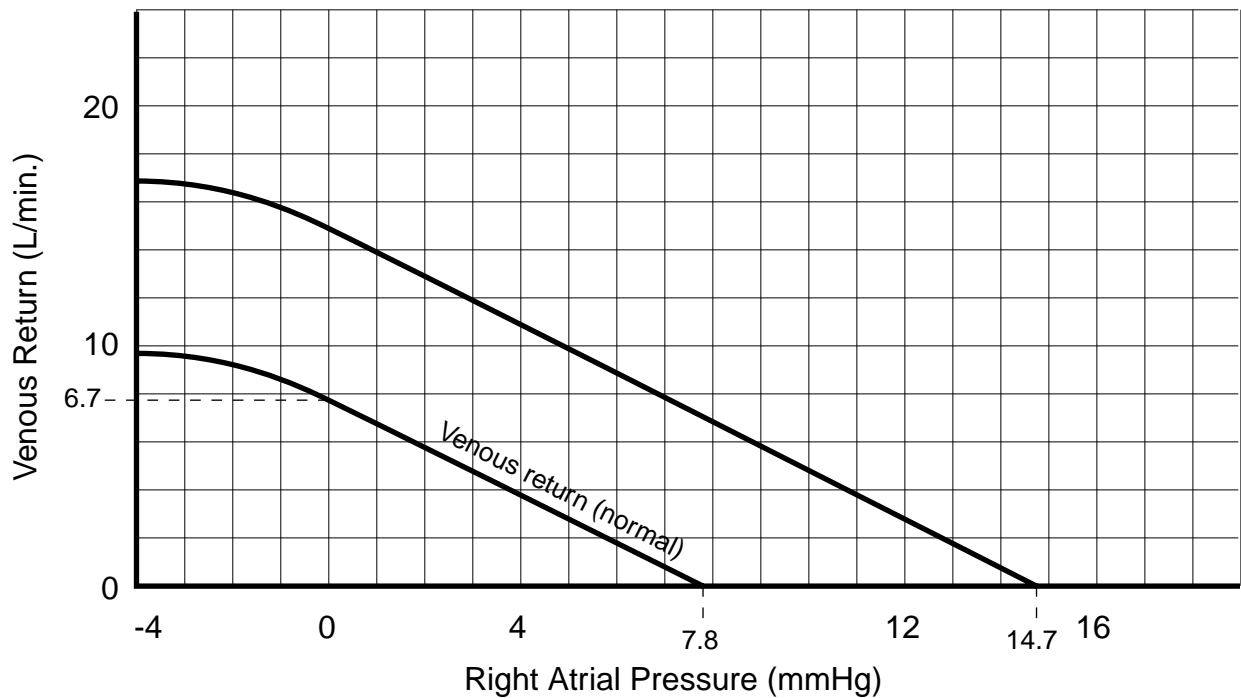
blood volume).



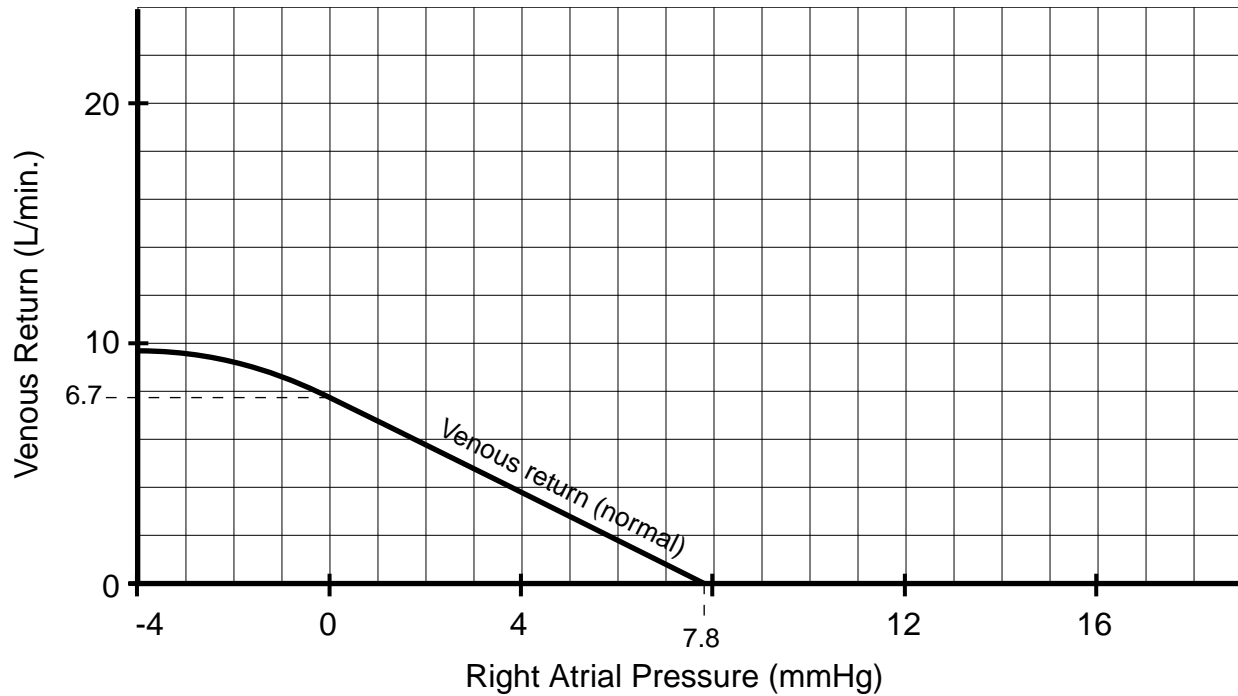
$$V_0 = 2500 \text{ ml}$$

$$P_{MS} = \frac{V_t - V_0}{C_a + C_v} = \frac{4000 \text{ mL} - 2500 \text{ mL}}{2 \text{ mL/mmHg} + 100 \text{ mL/mmHg}} = 14.7 \text{ mmHg}$$

P_{MS} increases so curve shifts to the right (x-intercept at 14.7 mmHg). No change in slope.



C. The arteriolar vessel radius decreases by 10%.



$$R \propto \frac{1}{r^4}$$

$$r_2 = 0.9r_1 \Rightarrow R_2 = \frac{R_1}{(0.9)^4} = 1.5R_1$$

$$R_1 = R_a = 1 \text{ mmHg} \cdot \text{s} / \text{mL}$$

$$R_2 = 1.5 \text{ mmHg} \cdot \text{s} / \text{mL}$$

$$Q_1 = \frac{P_{MS} - P_f}{R_v + R_a \frac{C_a}{C_a + C_v}}$$

$$= \frac{7.8 \text{ mmHg} - 0 \text{ mmHg}}{0.05 \text{ mmHg} \cdot \text{s} / \text{mL} + 1 \text{ mmHg} \cdot \text{s} / \text{mL} \left(\frac{2 \text{ mL/mmHg}}{2 \text{ mL/mmHg} + 100 \text{ mL/mmHg}} \right)}$$

$$\frac{112 \text{ mL}}{\text{sec}} = 6.7 \text{ L/min}$$

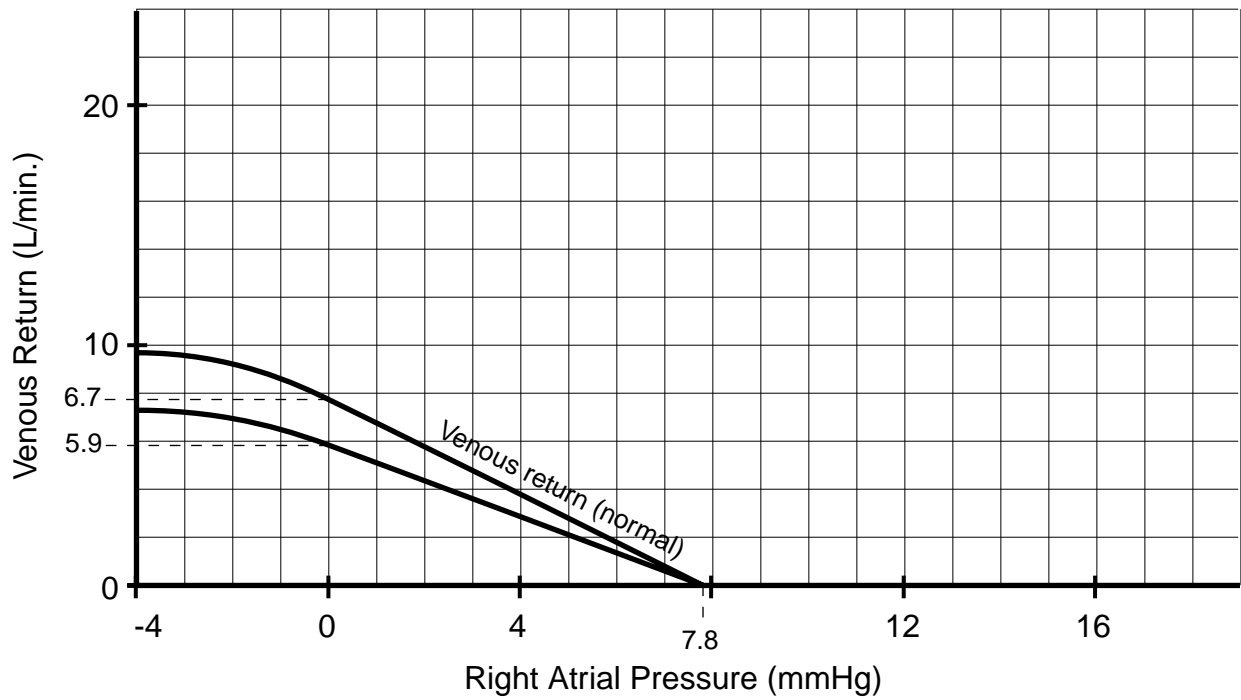
$$Q_2 = \frac{7.8 \text{ mmHg} - 0 \text{ mmHg}}{0.05 \text{ mmHg} \cdot \text{s} / \text{mL} + 1.5 \text{ mmHg} \cdot \text{s} / \text{mL} \left(\frac{2}{102} \right)}$$

$$= \frac{98.2 \text{ mL}}{\text{sec}}$$

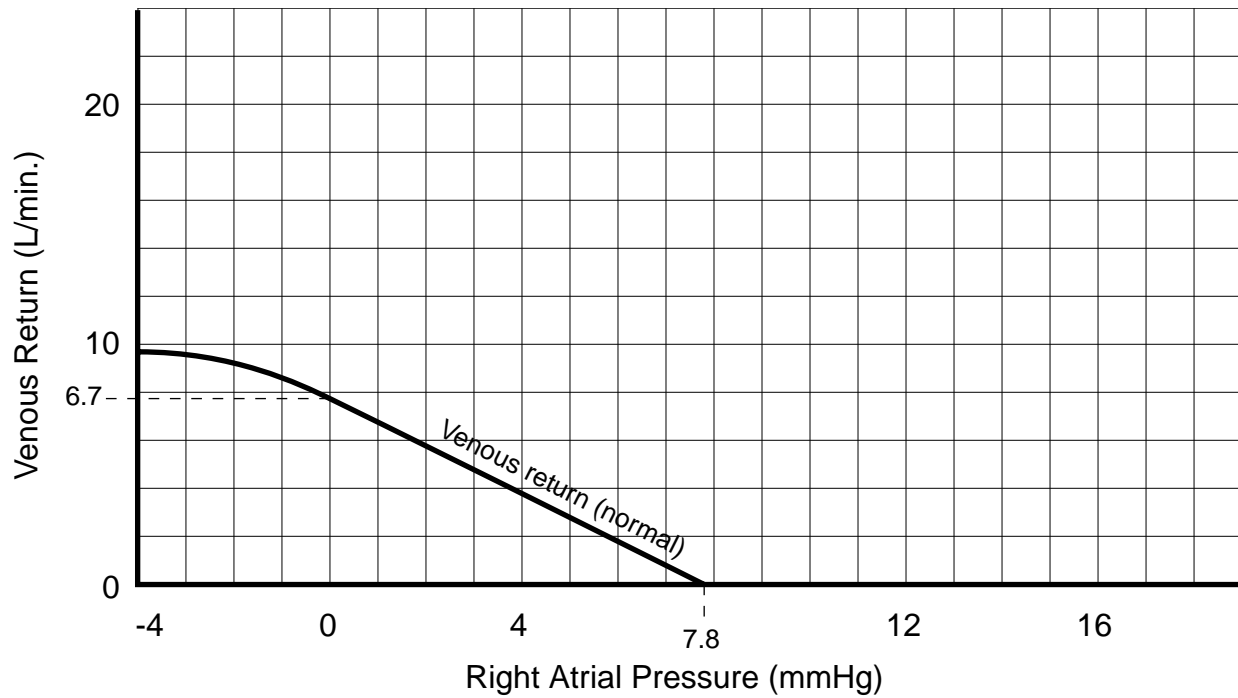
$$= 5.89 \text{ L/min}$$

$$\frac{Q_2}{Q_1} = \frac{5.9}{6.7} = 0.88$$

Slope decreased by 12%, x-intercept (P_{MS}) unchanged.



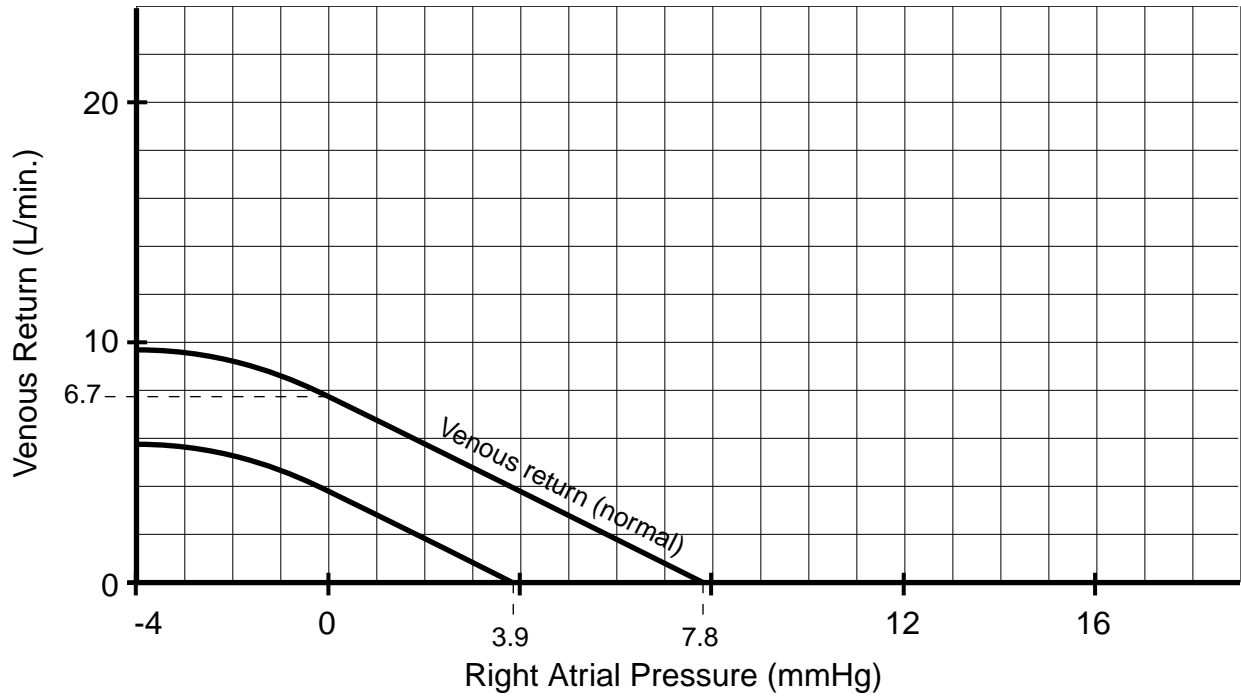
D. The patient develops a major hemorrhage, decreasing total volume to 3600 mL.



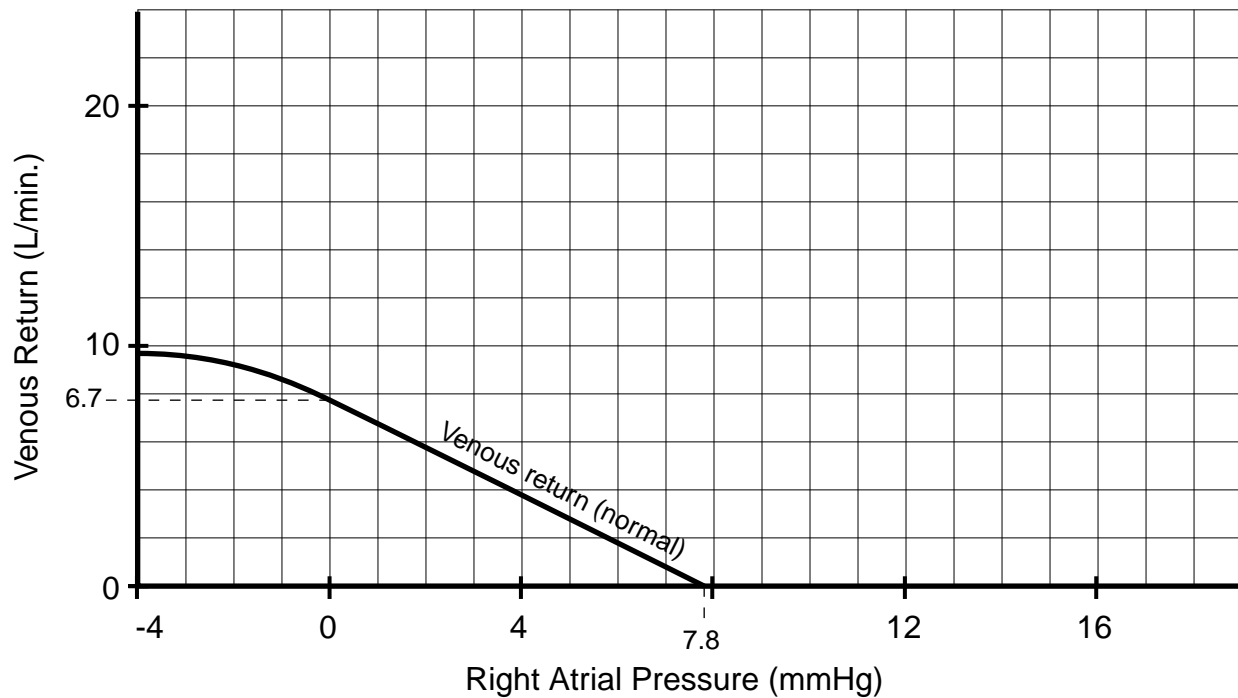
$$V_t = 3600 \text{ mL}$$

$$P_{MS} = \frac{3600 \text{ mL} - 3200 \text{ mL}}{2 \text{ mL/mmHg} + 100 \text{ mL/mmHg}} = 3.92 \text{ mmHg}$$

P_{MS} is half normal value (so is x -intercept), slope unchanged.



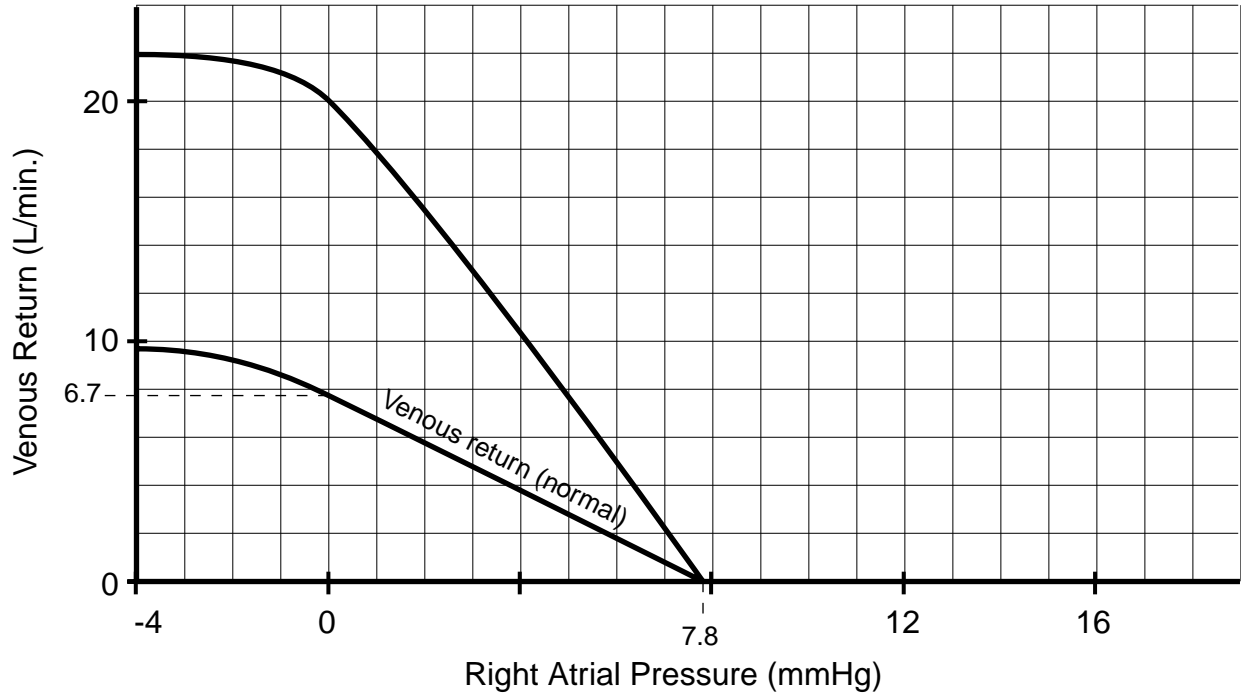
E. The patient develops a significant atriovenous fistula. (Use model from Problem 1.) Assume leg resistance is twice the fistula resistance.



From Problem 1:

$$\frac{Q_2}{Q_1} = 1 + \frac{R_L}{R_F} = 1 + 2 = 3$$

The slope is three times greater ($Q_2 \approx 20$ L/min). The x-intercept (P_{MS}) is unchanged.



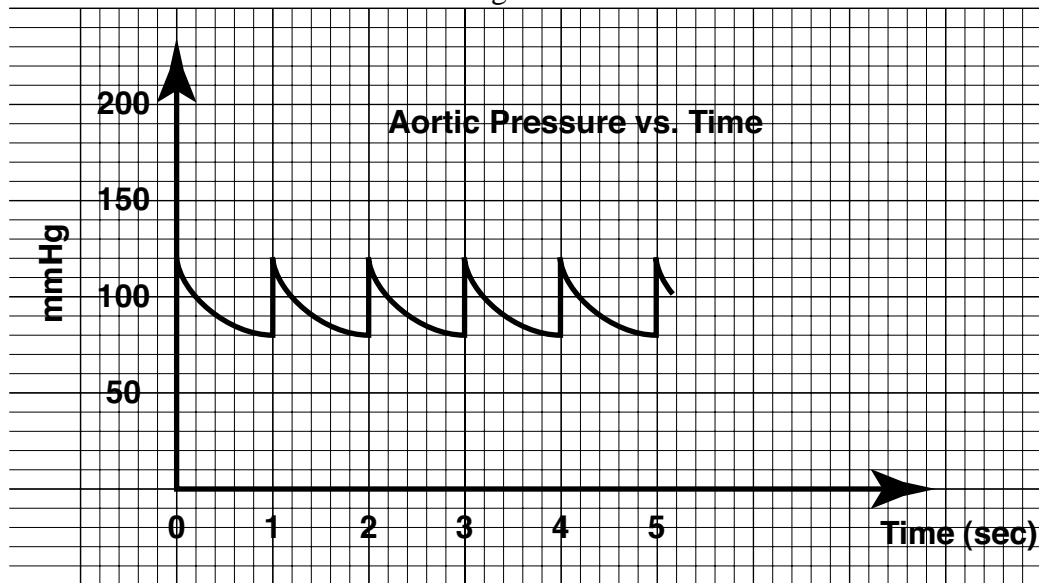
Problem 3

A patient developed complete heart block following corrective surgery for a congenital heart defect at age 25. He was given a permanent cardiac pacemaker which was set at a rate of 60 bpm. At the time of the pacemaker insertion a measurement of cardiac output was made using the Fick method. The data are given below:

Arterial O ₂ content	=	19.2 ccO ₂ per 100 cc blood
Mixed venous O ₂ content	=	15.0 ccO ₂ per 100 cc blood
Oxygen consumption	=	252 cc per minute
Oxygen capacity	=	20 cc O ₂ per 100 cc blood

Consider the idealized arterial pressure waveform as recorded from the aorta, is shown in Figure 2. Consider the Windkessel model of the peripheral circulation, driven by an idealized flow “impulse” source.

Figure 2:



- A. Calculate the cardiac output and stroke volume.

Cardiac output is volume of blood needed to account for oxygen uptake. If we do mass balance for oxygen before and after blood goes through the lung, we have:

$$\begin{aligned}
 O_2 + V_{vO_2} &= V_{aO_2} \\
 O_2 + Q \cdot C_{vO_2} &= Q \cdot C_{aO_2} \\
 Q &= \frac{O_2}{C_{aO_2} - C_{vO_2}}
 \end{aligned}$$

Arterial-venous difference in oxygenation is $(19.2 - 15.0) = 4.2$ cc O₂ / 100 cc blood:

$$\text{C.O.} = \frac{252 \text{ cc O}_2/\text{min}}{4.2 \text{ cc O}_2/100 \text{ cc blood}} = 6000 \text{ cc blood/min} = 6 \text{ L blood/min}$$

$$\text{S.V.} = \frac{6000 \text{ cc/min}}{60 \text{ beats/min}} = \boxed{100 \text{ cc/beat}}$$

- B. Calculate the peripheral resistance, R_a . Express answer in “peripheral resistance units” (PRU).

$$P_s = 120 \text{ mmHg} \quad P_d = 80 \text{ mmHg}$$

Use Windkessel model:

$$(i) \quad P(t) = P_s e^{-t/RC}$$

$$(ii) \quad P_d = P_s e^{-T/RC}$$

$$(iii) \quad \frac{RC}{T} = \frac{1}{\ln(P_s/P_d)}$$

$$\bar{P} = \frac{1}{T} \int_0^T P(t) dt = \frac{1}{T} \int_0^T P_s e^{-t/RC} dt = \left(\frac{RC}{T} \right) P_s (1 - e^{-T/RC})$$

Substituting in equations (ii) and (iii) into this expression for \bar{P} , we obtain

$$\bar{P} = \frac{P_s - P_d}{\ln(P_s/P_d)}$$

$$\bar{P} = \frac{(120 - 80) \text{ mmHg}}{\ln(120/80)} = 99 \text{ mmHg}$$

$$R = \frac{\bar{P}}{\text{C.O.}} = \frac{99 \text{ mmHg}}{100 \text{ cc/sec}} = \boxed{0.99 \text{ PRU}}$$

- C. The patient felt quite well during the following 6 months, and began participating in a regular exercise program. On one occasion after jogging his usual 2 miles, his blood pressure was found to be 150/70. Assuming no significant structural changes had occurred in his aorta, estimate his C.O.

$$C = \frac{\Delta V}{\Delta P}$$

$$((\text{C.O.})_2 = f_2 \Delta V_2 \quad \text{where } f = \text{heart rate, } \Delta V = \text{stroke volume}$$

$$= f_2 \Delta P_2 C$$

$$= f_2 \Delta P_2 \frac{\Delta V_1}{\Delta P_1} \quad \text{assuming } C \text{ unchanged}$$

$$= f_2 \Delta V_1 \frac{\Delta P_2}{\Delta P_1} \quad \text{where } \Delta P = \text{pulse pressure}$$

$$= (60/\text{min})(.1 \text{ L})(80/40)$$

$$= \boxed{12 \text{ L/min}}$$

- D. At age 60, he continued to be quite healthy, and his resting stroke volume had not changed. However, his arterial capacitance had decreased to 1/2 its original value as he had aged. Calculate his systolic and diastolic B.P. at rest assuming no change in peripheral resistance.

$$\begin{aligned}
 P_{s_2} - P_{d_2} &= \Delta P_2 \\
 \Delta P_2 &= \Delta V_2 / C_2 \\
 &= 2\Delta V_2 / C_1 \\
 &= 2\Delta V_1 / C_1 \\
 &= 2\Delta P_1 = 80 \text{ mmHg} \\
 \frac{P_{d_2}}{P_{s_2}} &= e^{-T/RC_2} \\
 &= e^{-2T/RC_1} \\
 &= (e^{-T/RC_1})^2 \\
 &= \left(\frac{P_{d_1}}{P_{s_1}}\right)^2 \\
 \Delta P_2 &= P_{s_2} - P_{d_2} \\
 &= P_{s_2} - P_{s_2} \left(\frac{P_{d_1}}{P_{s_1}}\right)^2 \\
 P_{s_2} &= \frac{\Delta P_2}{1 - \left(\frac{P_{d_1}}{P_{s_1}}\right)^2} \\
 &= \frac{80 \text{ mmHg}}{1 - (80/120)^2} \\
 P_{s_2} &= \boxed{144 \text{ mmHg}} \\
 P_{d_2} &= P_{s_2} - \Delta P_2 = \boxed{64 \text{ mmHg}}
 \end{aligned}$$

- E. At age 80, the patient's aortic capacitance had not changed from that at age 60 but his heart had begun to show signs of congestive heart failure. Since his pacemaker was rate-programmable (the pulse rate could be adjusted non-invasively over a wide range), his cardiologist wanted to select an optimal heart rate to maximize resting cardiac output. Could you suggest a simple technique to estimate cardiac output vs. pacemaker rate?

$$\begin{aligned}C.O. &= f \cdot \Delta V \\ \Delta V &= C \cdot \Delta P\end{aligned}$$

Assuming C is unchanged from part (D), measure ΔP using blood pressure cuff for each heart rate f and compute $C.O.$