Sets and Functions: Key Ideas

The notion of *set* is one of the central ideas of modern mathematics. I cannot define the notion for you. To define a notion, you explain it in more basic terms, and there the notion of *set* is rock-bottom basic. I can, however, give you some rough synonyms:

class

collection

totality

plurality

group

family

ensemble (in French)

Menge (in German)

One feature of the notion of set is that, whereas the members of a family are all related somehow, the members of a set need have nothing in common. Thus there is a set whose members are me, the number 19, Socrates' nose, and Mark Twain's sense of humor. Also, a set is determined by its elements. This distinguishes sets from properties. The property of being a creature with a heart is a different property from being a creature with kidneys, even if it turns out that exactly the same creatures have both. But if every creature with a heart has kidneys and *vice versa*, then the set of creatures with a heart and the set of creatures with kidneys are one and the same. Thus we have:

Axiom of Extensionality. For any sets A and B, $A = B$ if and only if A and B have exactly the same elements.

One way to name a set is by listing its elements, thus: {Mercury, Venus, Earth, Mars}. The axiom of extensionality tells us that, in naming a set by listing it, order and repetitions don't matter.

Another way to name a set is by giving some feature its elements share that nothing else has, thus: $\{x$: x is an inner planet$\}$. We use the symbol "$\in$" to indicate set membership, thus: Mercury $\in$ $\{x$: x is an inner planet$\}$ if and only if Mercury is an inner planet. As a general rule, we have this:

> For any a, $a \in \{x$: x is __________$\}$ if and only if a is __________.

This *naive comprehension principle* cannot be upheld in full generality, however. If it could, we would have this:

> For any a, $a \in \{x$: x is a set that isn't an element of itself$\}$ if and only if a is a set that isn't an element of itself.

Putting in "$\{x$: x is a set that isn't an element of itself$\}$" for "a," we get:

> $\{x$:x is a set that isn't an element of itself$\}$ $\in$ $\{x$:x is a set that isn't an element of itself$\}$ if and only if $\{x$: x is a set that isn't an element of itself].

which is absurd. This is *Russell's paradox*, discovered by Bertrand Russell in 1901. Figuring out a principled way to restrict the naive comprehension principle so as to avoid Russell's paradox while maintaining enough of the principle to satisfy the needs of mathematics was one of the central challenges of twentieth century mathematics. I hope we get a chance to talk about it, but not today. For now, let me ask you to take my word for it that the things we use here to fill in the blank in the naive comprehension principle are all legit.

For any sets A and B, $A \subseteq B$ if and only if every element of A is an element of B. We have:

If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

$A \subseteq A$.

$A = B$ if an only if $A \subseteq B$ and $B \subseteq A$.

$\varnothing$, the *null set* or the *empty set* is the set that has no elements. $\varnothing \subseteq A$, for every set A.

For any sets A and B, $A \cup B$, the *union* of A and B, = {x: $x \in A$ or $x \in B$ (or both)}. We have:

$A \cup B = B \cup A$..

$(A \cup B) \cup C = A \cup (B \cup C)A$

$A \cup A = A$.

$A \subseteq B$ if and only if $A \cup B = B$.

For any sets A and B, $A \cap B$, the *intersection* of A and B, = {x: $x \in A$ and $x \in B$}. We have:

$A \cap B = B \cap A$

$(\mathrm{A} \cap \mathrm{B}) \cap \mathrm{C} = \mathrm{A} \cap (\mathrm{B} \cap \mathrm{C})$

$\mathrm{A} \cap \mathrm{A} = \mathrm{A}$

$\mathrm{A} \subseteq \mathrm{B}$ if and only if $\mathrm{A} \cap \mathrm{B} = \mathrm{A}$.

$(\mathrm{A} \cap \mathrm{B}) \cup \mathrm{C} = (\mathrm{A} \cup \mathrm{C}) \cap (\mathrm{B} \cup \mathrm{C})$

$(\mathrm{A} \cup \mathrm{B}) \cap \mathrm{C} = (\mathrm{A} \cap \mathrm{C}) \cup (\mathrm{B} \cap \mathrm{C})$

For any sets A and B, the difference, $\mathrm{A} \sim \mathrm{B} = \{\mathrm{x}: \mathrm{x} \in \mathrm{A} \text{ and } \mathrm{x} \notin \mathrm{B}\}$. We have:

$\mathrm{A} \sim \mathrm{A} = \varnothing$

$(\mathrm{A} \cup \mathrm{B}) \sim \mathrm{C} = (\mathrm{A} \sim \mathrm{C}) \cup (\mathrm{B} \sim \mathrm{C})$

$\mathrm{A} \sim (\mathrm{B} \cup \mathrm{C}) = (\mathrm{A} \sim \mathrm{B}) \cap (\mathrm{A} \sim \mathrm{C})$

$(A \cap B) \sim C = (A \sim C) \cap (B \sim C)$

$A \sim (B \cap C) = (A \sim B) \cup (A \sim C)$

$(A \sim B) \sim C = A \sim (B \cup C)$

$A \sim (B \sim C) = (A \sim B) \cup (A \cap (B \cap C))$

$(A \sim B) \cup (A \cap B) = A.$

$(A \sim B) \cap (A \cap B) = \varnothing.$

$(A \sim B) \cup (B \sim A) = (A \cup B) \sim (A \cap B)$

$(A \sim B) \cap (B \sim A) = \varnothing.$

If $\mathscr{F}$ is a set of sets, $\bigcup\mathscr{F}$, the *union* of $\mathscr{F}$, is $\{x$: x is an element of at least one element of $\mathscr{F}\}$. Thus $A \cup B = \bigcup \{A, B\}$.

If $\mathscr{F}$ is a nonempty set of sets, $\bigcap\mathscr{F}$, the *intersection* of $\mathscr{F}$, is $\{x$: x is an element of every element of $\mathscr{F}\}$. Thus $A \cap B = \bigcap \{A, B\}$. We have, for $\mathscr{F}$ a nonempty set of sets and C a set:

$(\bigcup\mathscr{F}) \sim C = \bigcup \{A \sim C: A \in \mathscr{F}\}.$

$(\bigcap\mathscr{F}) \sim B = \bigcap \{A \sim C: A \in \mathscr{F}\}$

$C \sim (\bigcup\mathscr{F}) = \bigcap \{C \sim A: A \in \mathscr{F}\}$

$C \sim (\bigcap\mathscr{F}) = \bigcup (C \sim A: A \in \mathscr{F}\}.$

For given individuals a and b, the set $\{a,b\}$ is the *unordered pair* formed by a and b. It's unordered, because we have $\{a,b\} = \{b,a\}$. It's useful also to have a way of putting two things together where you keep track of the order. Thus, for given a and b, we form the unordered pair $\langle a,b\rangle$ so that (unless $a = b$), $\langle a,b\rangle$ will be different from $\langle b,a\rangle$. The fundamental principle governing ordered pairs is this:

Law of Ordered Pairs. For any a, b, c, and d, $<a,b> = <c,d>$ if and only if a = c and $b = d$.

The notion of ordered pair is especially useful in geometry, where we associate a point P with an ordered pair $<x,y>$, where x represents P's horizontal location and y represents its vertical location. More precisely x is ± the distance of P from the y-axis, positive if x is to the right of the axis, negative if it's to the left, 0 if its on the axis. y is ± the distance of P from the x-axis, positive if P is above the axis, negative if it's below, 0 if it's right on the axis. This association permitted a marriage of geometry and algebra that has been both happy and fruitful.

Set theorist's sometimes treat the notion ordered pairs as a defined notion, stipulating that $<a,b>$ is to be $\{\{a\}, \{a,b\}\}$. If you make that stipulation, you can derive the law of ordered pairs from the axiom of extensionality. Doing this has the advantage of reducing the number of primitive notions, so that you can build your theory on a very small number of basic ideas, treating all the other ideas as defined. For our purposes, however, the only thing we need to know about ordered pairs is the law of ordered pairs.

Now that we have ordered pairs, we want to go on to introduce ordered triples, ordered quadruples, and so on. We can enlist the notion of ordered pair to do all the work. The ordered triple $<a,b,c>$ is defined to be the ordered pair $<<a,b>,c>$. The ordered quadruple $<a,b,c,d>$ is defined as the ordered pair $<<a,b,c>,d>$, which is $<<<a,b>,c>,d>$. And so on. The ordered n+1-tuple $<a_1, a_2,..., a_n, a_{n+1}>$ is the ordered pair whose first component is the ordered n-tuple $<a_1, a_2,..., a_n>$ and whose second component is a_{n+1}. It will be convenient to fill things out by taking the 1-*tuple* $<a>$ to be a itself.

For sets A and B, the *Cartesian product* A × B is $\{<a,b>: a \in \mathrm{A} \text{ and } b \in \mathrm{B}\}$. A subset f of A × B is said to be a *function* from A to B (in symbols, f: A → B) just in case, for each element a of A there is exactly one element b of B with $<a,b> \in f$. You can think of f as a machine that, given an element of A as input, always gives an element of B as the output; the output is uniquely determined by the input. If f: A → B and $<a,b> \in f$, we write $f(a) = b$.

If f: A → B and g: B → C, the *composition* $g \circ f$ is the function: A → C given by $g \circ f(a) = g(f(a))$. You screw the output hose from f onto the input hose for g. If h: C → D, we have $(h \circ g) \circ f = h \circ (g \circ f)$.

A function f from A to B is said to be *one-one* or *injective* just in case, whenever we have $f(x) = f(y)$, we have $x = y$, so that you never get the same output for two different inputs.

If f is a function from A to B, A, the set of inputs, is said to be the *domain* of f, while the *range* of f is the set of all $b \in \mathrm{B}$ such that, for some a in A, $f(a) = b$; that is, the range of f is the set of outputs. If the range of f is all of B, f is said to be *onto* or *surjective.*

If f is both injective and surjective, it is said to be *bijective* or a *one-one correspondence*. In particular, the identity map on A, $\mathrm{ID}_\mathrm{A} = \{<a,a>: a \in \mathrm{A}\}$ is a bijection from A to itself. If f is a bijection: A → B, the *inverse*, f^{-1} is the bijection: B → A given by $f^{-1} = \{<b,a>: <a,b> \in f\}$. We have $f^{-1} \circ f = \mathrm{ID}_\mathrm{A}$ and $f \circ f^{-1} = \mathrm{ID}_\mathrm{B}$.

Where $\mathbb{R}$ is the set of real numbers, $\{<x, x^3>$: is a bijection: $\mathbb{R} \rightarrow \mathbb{R}$. The function $\{<x, x^3 - 6x^2 + 11x - 6: x \in \mathbb{R}\}$ is surjective but not one-one, since the inputs 1, 2, and 3 all give the output 0. The arctangent function, $\{<x,y>: -{}^{\pi}/_2 < y < {}^{\pi}/_2$ and $x = \tan y\}$, is injective but not surjective; it's an increasing function whose range is the set of numbers between $-{}^{\pi}/_2$ and ${}^{\pi}/_2$. The function $\{<x, x^2>: x \in \mathbb{R}\}$ is neither injective nor surjective. It isn't injective because it gives the output 4 for both the input 2 and the input -2., and it isn't surjective because none of the negative numbers are in its range.

Let B be the set {Al, Bruce}, and let G be the set {Xochitl, Yolanda, Zaida}. Then B × B is the 4-element set {<a,a>, <a,b>, <b,a>, <b,b>} (using "a" and "b" to abbreviate "Al" and "Bruce"). B × G is the 6-element set {<a,x>, <a,y>, <a,z>, <b,x>, <b,y>, <b,z>}. G × B is the 6-element set {<x,a>, <x,b>, <y,a>, <y,b>, <z,a>, <z,b>}. G × G is the 9-element set {<x,x>, <x,y>, <x,z>, <y,x>, <y,y>,<y,z>, <z,x>, <z,y>, <z,z>}.

There are four functions from B to B, namely:

{<a,a>, <b,a>}

{<a,a>, <b,b>}

{<a,b>, <b,a>}

{<a,b>, <b,b>}

The second and third are bijective, whereas the other two are neither injective nor surjective.

There are nine functions from B to G, namely:

{<a,x>, <b,x>}

{<a,x>, <b,y>}

{<a,x>, <b,z>}

{<a,y>, <b,x>}

{<a,y>, <b,y>}

{<a,y>, <b,z>}

{<a,z>, <b,x>}

{<a,z>, <b,y>}

{<a,z>, <b,z>}

Of these, the second, third, fourth, sixth, seventh, and eighth are injective. None of the functions are surjective.

There are eight function from G to B, namely:

{<x,a>, <y,a> <z,a>}

{<x,a>, <y,a>, <z,b>}

{<x,a>, <y,b>, <z,a>}

{<x,a>, <y,b>, <z,b>}

{<x,b>, <y,a>, <z,a>}

{<x,b>, <y,a>, <z,b>}

{<x,b>, <y,b>, <z,a>}

{<x,b>, <y,b>, <z,b>}

Of these eight functions, all but the first and last are surjective. None are injective.

There are twenty-seven functions from G to G, namely:

{<x,x>, <y,x>, <z,x>}   neither

{<x,x>, <y,x>, <z,y>}	neither
{<x,x>, <y,x>, <z,z>}	neither
{<x,x>, <y,y>, <z,x>}	neither
{<x,x>, <y,y>,<z,y>}	neither
{<x,x>, <y,y>, <z,z>}	bijective
{<x,x>, <y,z>, <z,x>}	neither
{<x,x>, <y,z>, <z,y>}	bijective
{<x,x>, <y,z>, <z,z>}	neither
{<x,y>, <y,x>, <z,x>}	neither
{<x,y>, <y,x>, <z,y>}	neither
{<x,y>, <y,x>, <z,z>}	bijective
{<x,y>, <y,y>, <z,x>}	neither
{<x,y>, <y,y>,<z,y>}	neither
{<x,y>, <y,y>, <z,z>}	neither
{<x,y>, <y,z>, <z,x>}	bijective
{<x,y>, <y,z>, <z,y>}	neither
{<x,y>, <y,z>, <z,z>}	neither
{<x,z>, <y,x>, <z,x>}	neither
{<x,z>, <y,x>, <z,y>}	bijective
{<x,z>, <y,x>, <z,z>}	neither
{<x,z>, <y,y>, <z,x>}	bijective
{<x,z>, <y,y>,<z,y>}	neither

{<x,z>, <y,y>, <z,z>}	neither
{<x,z>, <y,z>, <z,x>}	neither
{<x,z>, <y,z>, <z,y>}	neither
{<x,z>, <y,z>, <z,z>}	neither