Function Signs

We are going to liberalize the predicate calculus a little bit by introducing notations for functions. An *n-ary function* on a set A is a set of n+1-tuples of elements of A that satisfies this condition:

> For each n-tuple $<a_1,a_2,...,a_n>$ of elements of A, there is one and only one element b of A such that the n+1-tuple $<a_1,a_2,...,a_n,b>$ is an element of the set.

If f is an n-ary function, we write "$f(a_1,a_2,...,a_n) = b$" to indicate that $<a_1,a_2,...,a_n,b> \in f$.

A language for the *predicate calculus with identity and function signs* will have as its nonlogical symbols individual constants, predicates, and — this is new — n-ary function signs, for n a positive integer. The logical symbols will be the variables, the connectives ("$\vee$," "$\wedge$," "$\rightarrow$," "$\leftrightarrow$," and "$\neg$) and quantifiers ("$\forall$" and "$\exists$"), the punctuation marks ("(," "")," and ","), and the predicate "=." "=" is considered a logical symbol, because it has a fixed semantic role determined by the definition of "interpretation," rather than having a semantic role that varies from one interpretation to another.

The *terms* constitute the smallest class of expressions that meets the following conditions:

> Every variable is a term.
> Every individual constant is a term.
> If $\tau_1, \tau_2,..., \tau_n$ are terms and f is an n-ary function signs, $f(\tau_1,\tau_2,...,\tau_n)$ is a term.

A term with no variables is *closed.*

If $\mathcal{A}$ **is to be an interpretation of such a language, it needs to assign a value to "$\forall$," to each of the individual constants, and to each of the predicates, just as before. It also needs to assign a value to each of the function signs, subject to the following constraint:**

> If f is an n-ary function sign, $\mathcal{A}$**(f), also written $f^{\mathcal{A}}$, is an n-ary function on $|\mathcal{A}|$.**

The semantic role of the function signs is given by the following:

> **Definition.** If $\mathcal{A}$ **is an interpretation and σ is a variable assignment for $\mathcal{A}$, the *denotation* of a term τ, $Den_{\sigma,\mathcal{A}}$, is defined as follows:**
>
> **If τ is a variable, $Den_{\sigma,\mathcal{A}}(\tau) = \sigma(\tau)$.**
> **If τ is an individual constant, $Den_{\sigma,\mathcal{A}}(\tau) = \mathcal{A}(\tau)$.**
> **If τ has the form $f(\rho_1,\rho_2,...,\rho_n)$, $Den_{\sigma,\mathcal{A}}(\tau) = f^{\mathcal{A}}(Den_{\sigma,\mathcal{A}}(\rho_1),Den_{\sigma,\mathcal{A}}(\rho_2),...,Den_{\sigma,\mathcal{A}}(\rho_n))$.**

If φ is an atomic formula of the form $R\tau_1\tau_2...\tau_n$, σ satisfies φ under $\mathcal{A}$ **just in case the n-tuple $\langle \mathrm{Den}_{\sigma,\mathcal{A}}(\tau_1),\mathrm{Den}_{\sigma,\mathcal{A}}(\tau_2),...,\mathrm{Den}_{\sigma,\mathcal{A}}(\tau_n)\rangle$ is an element of $R^{\mathcal{A}}$. With this adaptation, the definition of satisfaction under an interpretation is unchanged.**

The introduction of function signs is gratuitous. Instead of using an n-ary function sign f, we could just as well introduce an n+1-place predicate F. Then, instead of writing "$f(v_1,v_2,...,v_n)$," we could write "$(\iota v_{n+1})F(v_1,v_2,...,v_n,v_{n+1})$."

The rules of inference are unchanged, except that, here and there, they'll say "closed term" instead of "individual constant." Let me write them out in full, for reference:

PI At any stage of a derivation, you may write down a sentence φ with any set that contains φ as its premise set.

TC If you have written down sentences ψ_1, ψ_2,..., ψ_n in a derivation, and φ is a tautological consequence of $\{\psi_1,\psi_2,...,\psi_n\}$, then you may write down sentence ψ, taking the premise set to be the union of the premise sets of the ψ_is. In particular, if φ is a tautology, we can write φ with the empty premise set.

CP If you have derived ψ with premise set $\Gamma \cup \{\varphi\}$, you may write $(\varphi \rightarrow \psi)$ with premise set Γ.

US If you've derived $(\forall v)\varphi$, you may derive $\varphi^{v}/_{\tau}$ with the same premise set, for any variable v and closed term τ.

UG For any variable v, if you've derived $\varphi^{v}/_{c}$ from Γ and if the individual constant c doesn't appear in φ or in any of the sentences in Γ, you may derive $(\forall v)\varphi$ with premise set Γ

QE From $\neg(\forall v)\neg\varphi$, you may infer $(\exists v)\varphi$ with the same premise set, and vice versa, for each variable v.
From $(\forall v)\neg\varphi$, you may infer $\neg(\exists v)\varphi$ with the same premise set, and vice versa, for each variable v.
From $\neg(\forall v)\varphi$, you may infer $(\exists v)\neg\varphi$ with the same premise set, and vice versa, for each variable v.
From $(\forall v)\varphi$, you may infer $\neg(\exists v)\neg\varphi$ with the same premise set, and vice versa, for each variable v.

EG If you have written $\varphi^{v}/_{\tau}$, for any variable v and closed term τ, you may write $(\exists v)\varphi$ with the same premise set.

ES Suppose that you have derived $(\exists v)\varphi$ with premise set Δ and that you have derived ψ with premise set $\Gamma \cup \{\varphi^{v}/_{c}\}$, for some individual constant c and variable v. Suppose

further that the constant c does not appear in φ, in ψ, or in any member of Γ. Then you may derive ψ with premise set $\Delta \cup \Gamma$.

IR You may write any sentence of the form τ=τ, with the empty set of premisses.

SI If you've written either τ=ρ or ρ=τ with premiss set Γ and you've written $\varphi^{v}/_{\tau}$ with premiss set Δ, you may write $\varphi^{v}/_{\rho}$ with premiss set $\Gamma\cup\Delta$.

As an example, let's derive "(∃x)(∃y)(∃z)(¬x=y ∧ (¬x=z ∧ ¬y=z))" from the premiss "(¬(∃x)s(x) = 0 ∧ (∀x)(∀y)(s(x) = s(y) → x = y))." The conclusion says that there are at least three individuals. In the same way, we can use the same premiss to derive the sentence that says "There are at least four individuals," the sentence that says "There are at least five individuals," and so on. The premiss is consistent — it's made true by taking "s" to denote the successor function on the natural numbers — but it isn't true in any finite model.

1	**1 (¬(∃x)s(x) = 0 ∧ (∀x)(∀y)(s(x) = s(y) → x = y))**	**PI**
1	**2 ¬(∃x)s(x) = 0**	**TC, 1**
1	**3 (∀x)(∀y)(s(x) = s(y) → x = y)**	**TC, 1**
1	**4 (∀x)¬s(x) = 0**	**QE, 2**
1	**5 ¬s(0) = 0**	**US, 4**
1	**6 ¬s(s(0)) = 0**	**US, 4**
1	**7 (∀y)(s(s(0)) = s(y) → s(0) = y)**	**US, 3**
1	**8 (s(s(0)) = s(0) → s(0) = 0)**	**US, 7**
1	**9 ¬s(s(0)) = s(0)**	**TC, 5, 8**
1	**10 (¬s(s(0)) = s(0) ∧ (¬s(s(0)) = 0 ∧ ¬s(0) = 0))**	**TC, 5, 6, 9**
1	**11 (∃z)(¬s(s(0)) = s(0) ∧ (¬s(s(0)) = z ∧ ¬s(0) = z))**	**EG, 10**
1	**12 (∃y)(∃z)(¬s(s(0)) = y ∧ (¬s(s(0)) = z ∧ ¬y = z))**	**EG, 11**
1	**13 (∃x)(∃y)(∃z)(¬x = y ∧ (¬x=z ∧ ¬y = z))**	**EG, 12**

As a second example, let's formalize the following inference:

(∀x)(∀y)(∀z)p(x,p(y,z)) = p(p(x,y),z)
(∀x)p(x,e) = x
(∀x)p(x,i(x)) = e
∴ (∀x)(∀y)(∀z)(p(x,z) = p(y,z) → x = y)

Disguised in a quirky notation,[*] this is the derivation in the theory of groups of the cancellation law:

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[*] The reason for the quaint notation is this: When we're using a 2-ary function sign, like "+" or "·," it's convenient to write the function sign between the terms it applies to, writing "x+y" or "x·y" instead of "+(x,y)" or "·(x,y)." But such a notation doesn't generalize to 3- or more-ary function signs. In practice, unless there's risk of confusion,

$(x \cdot z = y \cdot x \rightarrow x = y)$.

1	**1 $(\forall x)(\forall y)(\forall z)p(x,p(y,z)) = p(p(x,y),z)$**	**PI**
2	**2 $(\forall x)p(x,e) = x$**	**PI**
3	**3 $(\forall x)p(x,i(x)) = e$**	**PI**
4	**4 $p(a,c) = p(b,c)$**	**PI**
2	**5 $p(a,e) = a$**	**US, 2**
3	**6 $p(c,i(c)) = e$**	**US, 3**
2,3	**7 $p(a,p(c,i(c))) = a$**	**SI, 5, 6**

Applying US to line 1 three times, we continue:

1	**10 $p(a,p(c,i(c))) = p(p(a,c),i(c))$**	**US×3, 1**
1,2,3	**11 $p(p(a,c),i(c)) = a$**	**SI, 7, 10**
1,2,3,4	**12 $p(p(b,c),i(c)) = a$**	**SI, 4, 11**

Again, we apply US to line 1 three times:

1	**15 $p(b,p(c,i(c))) = p(p(b,c),i(c))$**	**US×3, 1**
1,2,3,4	**16 $p(b,p(c,i(c))) = a$**	**SI, 12, 15**
1,2,3,4	**17 $p(b,e) = a$**	**SI, 6, 16**
2	**18 $p(b,e) = b$**	**US, 2**
1,2,3,4	**19 $a = b$**	**SI, 17, 18**
1,2,3	**20 $(p(a,c) = p(b,c) \rightarrow a = b)$**	**CP, 4, 20**

We apply UG thrice to complete the proof:

1,2,3	**23 $(\forall x)(\forall y)(\forall z)(p(x,z) = p(y,z) \rightarrow x = y)$**	**UG×3, 20**

Notice what rule UG says is that, if you've derived $\varphi^{v}/_{c}$ from Γ and the *individual constant* c doesn't appear in φ or in any of the sentences in Γ, you may derive $(\forall v)\varphi$ from Γ. It doesn't say that, if you've derived $\varphi^{v}/_{\tau}$ from Γ and the *closed term* τ doesn't appear in φ or in any of the sentences in Γ, you may derive $(\forall v)\varphi$ from Γ. The latter rule would give us an unsound system, since it would enable us to derive the invalid sentence "$(\forall x)(\forall y)x=y$" from the empty set, as follows:

1	**1 $a=b$**	**PI**

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there's no need to insist on putting the function signs in front.

	2 f(a) = f(a)	**IR**
1	**3 f(a) = f(b)**	**SI, 1, 2**
1	**4 (∀x)x = f(b)**	**Illegitimate use of UG**
1	**5 c = f(b)**	**US, 4**
1	**6 d = f(b)**	**US, 4**
1	**7 c = d**	**SI, 5, 6**
1	**8 (∀y)c=y**	**UG, 7**
1	**9 (∀x)(∀y)x=y**	**UG, 8**
	10 (a=b → (∀x)(∀y)x=y)	CP, 1, 9
	11 (∀z)(a=z → (∀x)(∀y)x=y)	UG, 10
	12 (a=a → (∀x)(∀y)x=y)	US, 11
	13 a=a	IR
	14 (∀x)(∀y)x=y	

If we look back at our original proof that UG is sound, we can see what's gone wrong. The proof makes use of the fact that, for any interpretation $\mathcal{A}$ **and constant c, it is possible to change what c denotes while leaving everything else unchanged.** The same doesn't hold for an arbitrary closed term. When we change what "f(a)" denotes, we'll have to change either what "a" denotes or what "f" denotes.

Similarly, we would get an unsound system if we changed "individual constant" to "closed term" in rule ES.

The proof of the completeness theorem is scarcely changed. We define a *Henkin set* to be a d-consistent set Γ of sentences such that, for each sentence, either the sentence or its negation is in Γ, and such that, whenever a existential sentence (∃v)φ is in Γ, there is an individual constant c such that $\varphi^{v}/_{c}$ is in Γ. Given a d-consistent set of sentences Δ, we first add infinitely many new constants to the language, then we show how to form, within the expanded language, a Henkin set that contains Δ. The proof is the same as before.

The proof that, given a Henkin set Γ, there is an interpretation under which all and only the sentences in Γ are true is only slightly different from before. The slight changes are to ensure that "=" behaves as intended. First, enumerate the individual constants in the language as c_0, c_1, c_2, c_3,.... Next define, for each j,

$c_j^{\mathcal{A}}$ **= the least natura**l number i such that the sentence $c_i = c_j$ is in Γ.

Define

$|\mathcal{A}|$ = {$c_j^{\mathcal{A}}$: j a natural number}.
<$j_1,j_2,\ldots,j_n$> is in $R^{\mathcal{A}}$ iff $Rc_{j_1}c_{j_2}\ldots c_{j_n}$ is in Γ

$f^{\mathcal{A}}(j_1,j_2,\ldots,j_n)$ = the least i such that c_i = f©$_{j_1}$,c_{j_2},...,c_{j_n}) is in Γ.

It is straightforward to verify that, "="$^{\mathcal{A}}$ **is the identity relation on** $|\mathcal{A}|$**; that, for any closed term** τ, $\tau^{\mathcal{A}}$ **= the least i such that** $c_i = \tau$ **is an element of** Γ**; and that, for any sentence** φ, φ **is true in** $\mathcal{A}$ **iff** $\varphi \in \Gamma$**.**

The derivation of the Compactness Theorem from Strong Completeness is the same as before.

The original version of the Löwenheim-Skolem theorem told us that, for any consistent set of sentences Γ**, there is a model whose domain is the set of natural numbers in which all the members of** Γ **are true. This result will no longer hold when we introduce identity into the language. "(∀x)(∀y)x=y" is consistent, but it isn't true under any interpretation whose domain is the set of natural numbers. Indeed, it isn't true** under any interpretation whose domain contains more than one element. We do, however, have this:

> **Löwenheim-Skolem Theorem.** For any consistent set of sentence Γ, there is an interpretation under which all the members of Γ are true whose domain consists entirely of natural numbers. Not every natural number has to appear in the domain.