Question 1

The trickiest things about this question is just stating what you want to prove in the right way.

Let $\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q_n}$ be atomic sentences. Let $v(\mathbf{R})$ be the truth value assigned to $\mathbf{R}$ by truth-value assignment v. Let $\ulcorner(\ldots(\mathbf{P}_1\&\mathbf{P}_2)\&\ldots\&\mathbf{P_n})\urcorner$ be the sentence such that $\mathbf{P_i} = \mathbf{Q_i}$ iff $v(\mathbf{Q_i}) = T$, $\mathbf{P_i} = \ulcorner\sim\mathbf{Q_i}\urcorner$ iff $v(\mathbf{Q_i}) = F$.

Showing that a sentence constructed in accordance with the characteristic sentence algorithm in TLB is, indeed, a characteristic sentence for the row of the truth-function schema in question is equivalent to showing that a truth-value assignment makes $\ulcorner(\ldots(\mathbf{P}_1\&\mathbf{P}_2)\&\ldots\&\mathbf{P_n})\urcorner$ true if and only if that truth-value assignment assigns truth-values to the relevant atomic sentences in the way that v does. (You should think of v as the truth-value assignment that assigns truth-values in the way indicated by the row of the truth-function schema in question.) Another way of saying this: $u(\ulcorner(\ldots(\mathbf{P}_1\&\mathbf{P}_2)\&\ldots\&\mathbf{P_n})\urcorner) = T$ iff $u(\mathbf{Q_i}) = v(\mathbf{Q_i})$, for all $\mathbf{i}$.

So what we want to show is that $u(\ulcorner(\ldots(\mathbf{P}_1\&\mathbf{P}_2)\&\ldots\&\mathbf{P_n})\urcorner) = T$ iff $u(\mathbf{Q_i}) = v(\mathbf{Q_i})$, for all $\mathbf{i}$.

Here's a proof:

1. By the fact in 6.1E (1d), $u(\ulcorner(\ldots(\mathbf{P}_1\&\mathbf{P}_2)\&\ldots\&\mathbf{P_n})\urcorner) = T$ iff $u(\mathbf{P}_1) = u(\mathbf{P}_2) = \ldots = u(\mathbf{P_n}) = T$.

2. $u(\mathbf{P}_1) = u(\mathbf{P}_2) = \ldots = u(\mathbf{P_n}) = T$ iff both

   (a) for all $\mathbf{i}$ such that $\mathbf{P_i} = \mathbf{Q_i}$, $u(\mathbf{P_i}) = T$, and

   (b) for all $\mathbf{i}$ such that $\mathbf{P_i} = \ulcorner\sim\mathbf{Q_i}\urcorner$, $u(\mathbf{P_i}) = T$

   (as all $\mathbf{i}$ are either such that $\mathbf{P_i} = \mathbf{Q_i}$, or such that $\mathbf{P_i} = \ulcorner\sim\mathbf{Q_i}\urcorner$).

3. Now, 2(a) is true iff for all $\mathbf{P_i} = \mathbf{Q_i}$, $u(\mathbf{Q_i}) = T$, obviously, and 2(b) is true iff for all $\mathbf{i}$ such that $\mathbf{P_i} = \ulcorner\sim\mathbf{Q_i}\urcorner$, $u(\mathbf{Q_i}) = F$ (by the definition of '$\sim$').

4. So $u(\mathbf{P}_1) = u(\mathbf{P}_2) = \ldots = u(\mathbf{P_n}) = T$ iff both

   (a) for all $\mathbf{P_i} = \mathbf{Q_i}$, $u(\mathbf{Q_i}) = T$

   (b) for all $\mathbf{i}$ such that $\mathbf{P_i} = \ulcorner\sim\mathbf{Q_i}\urcorner$, $u(\mathbf{Q_i}) = F$.

   (by 2, 3).

5. But for all $\mathbf{i}$ such that $\mathbf{P_i} = \mathbf{Q_i}$, $v(\mathbf{Q_1}) = T$, and for all i such that $\mathbf{P_i} = \ulcorner\sim\mathbf{Q_i}\urcorner$, $v(\mathbf{Q_i}) = F$ (by the definition of v above).

6. So $u(\mathbf{P}_1) = u(\mathbf{P}_2) = \ldots = u(\mathbf{P_n}) = T$ iff, $u(\mathbf{Q_i}) = v(\mathbf{Q_i})$, for all $\mathbf{i}$ (from 4, 5).

7. So $u(\ulcorner(\ldots(\mathbf{P}_1\&\mathbf{P}_2)\&\ldots\&\mathbf{P_n})\urcorner) = T$ iff $u(\mathbf{Q_i}) = v(\mathbf{Q_i})$ (from 1, 6).

Q.E.D.

Question 2

Let Γ be the set of atomic sentences.

Let L_1 be the set of all sentences $\mathbf{P}$ such that $\mathbf{P} \in L_1$ iff

(a) $\mathbf{P} \in \Gamma$, or

(b) $\mathbf{P}$ is of one of the following forms:

(i) $\ulcorner \sim \mathbf{Q} \urcorner$;

(ii) $\ulcorner \mathbf{Q} \& \mathbf{R} \urcorner$;

(iii) $\ulcorner \mathbf{Q} \vee \mathbf{R} \urcorner$;

where $\mathbf{Q}, \mathbf{R} \in L_1$.

Let L_2 be the set of all sentences $\mathbf{P}$ such that

(a) $\mathbf{P} \in \Gamma$, or

(b) $\mathbf{P}$ is of the form $\ulcorner \mathbf{Q} \downarrow \mathbf{R} \urcorner$, where $\mathbf{Q}, \mathbf{R} \in L_2$.

We know that {'$\sim$','&','$\vee$'} is TF-complete, so we know that for every truth-function, there is a sentence in L_1 that expresses that truth-function. If, for every sentence $\mathbf{P} \in L_1$, there is a sentence in L_2 that expresses the same truth function as $\mathbf{P}$, then it follows that for every truth-function, there is a sentence in L_2 that expresses that truth-function, from which it follows that {'$\downarrow$'} is TF-complete.

So, all that remains to be done to prove that {'$\downarrow$'} is TF-complete is to prove that for every sentence $\mathbf{P} \in L_1$, there is a sentence in L_2 that expresses the same truth function as $\mathbf{P}$.

I prove this by mathematical induction.

Basis Clause: For every atomic sentence $\mathbf{P} \in L_1$ there is a sentence in L_2 that expresses the same truth function as $\mathbf{P}$.

Inductive Step: If, for every sentence $\mathbf{P} \in L_1$ containing n or fewer connectives there is a sentence in L_2 that expresses the same truth-function as $\mathbf{P}$, then for every sentence $\mathbf{Q} \in L_1$ containing $n+1$ connectives there is a sentence in L_2 that expresses the same truth-function as L_2.

Clearly, it follows from Basis Clause and Inductive Step that for every sentence $\mathbf{P} \in L_1$, there is a sentence in L_2 that expresses the same truth function as $\mathbf{P}$.

The proof of Basis Clause is immediate; for every atomic sentence $\mathbf{P} \in L_1$ there is a sentence in L_2 that expresses the same truth function as $\mathbf{P}$ — namely, $\mathbf{P}$. (Besides, every atomic sentence expresses the same truth-function: the one that maps true to true and false to false.)

Now for Inductive Step. Consider an arbitrary sentence $\mathbf{Q} \in L_1$ containing $n+1$ connectives. $\mathbf{Q}$ is either of the form

(a) $\ulcorner\sim \mathbf{R}\urcorner$, or

(b) $\ulcorner\mathbf{R}\&\mathbf{S}\urcorner$, or

(c) $\ulcorner\mathbf{R} \vee \mathbf{S}\urcorner$.

I prove Inductive Step for each case in turn.

Case (a):

1. Suppose that, for every sentence $\mathbf{P} \in L_1$ containing n or fewer connectives there is a sentence in L_2 that expresses the same truth-function as $\mathbf{P}$ (i.e., suppose the antecedent of Inductive Step).

2. Then there is a sentence in L_2 that expresses the same truth-function as $\mathbf{R}$ (as $\mathbf{R}$ contains n connectives). Call that sentence in L_2 $\mathbf{T}$.

3. I claim that $\ulcorner\mathbf{T} \downarrow \mathbf{T}\urcorner$ expresses the same truth-funciton as $\ulcorner\sim \mathbf{R}\urcorner$. Here's a sub-proof of my claim:

   (i) A truth-value assignment makes $\ulcorner\sim \mathbf{R}\urcorner$ true iff it makes $\mathbf{R}$ false (by defintion of '$\sim$').

   (ii) A truth-value assignment makes $\mathbf{R}$ false iff it makes $\mathbf{T}$ false (by 2).

   (iii) A truth-value assignment makes $\mathbf{T}$ false iff it makes $\ulcorner\mathbf{T} \downarrow \mathbf{T}\urcorner$ true (by the definition of '$\downarrow$'. I leave the verification of this as an exercise for the reader).

   (iv) So a truth-value assignment makes $\ulcorner\sim \mathbf{R}\urcorner$ true iff it makes $\ulcorner\mathbf{T} \downarrow \mathbf{T}\urcorner$ true.

   (v) So $\ulcorner\sim \mathbf{R}\urcorner$ expresses the same truth-function as $\ulcorner\mathbf{T} \downarrow \mathbf{T}\urcorner$.

4. $\ulcorner\mathbf{T} \downarrow \mathbf{T}\urcorner \in L_2$ (by the definition of L_2).

5. So there exists a sentence of L_2 that expresses the same truth-function as $\ulcorner\sim \mathbf{R}\urcorner$.

6. So, assuming the antecedent of Inductive Step, for every sentence of the form $\ulcorner\sim \mathbf{R}\urcorner$, where $\mathbf{R}$ has n connectives, there is a sentence in L_2 that expresses the same truth-function as $\ulcorner\sim \mathbf{R}\urcorner$.

That proves Inductive Step for case (a).

Case (b):

1. Suppose that, for every sentence $\mathbf{P} \in L_1$ containing n or fewer connectives there is a sentence in L_2 that expresses the same truth-function as $\mathbf{P}$ (i.e., suppose the antecedent of Inductive Step).

2. Then there is a sentence in L_2 that expresses the same truth-function as $\mathbf{R}$ and a sentence of L_2 that expresses the same truth-function as $\mathbf{S}$ (as both $\mathbf{R}$ and $\mathbf{S}$ contain n or fewer connectives). Call those sentence in L_2 $\mathbf{T}$ and $\mathbf{U}$, respectively.

3. I claim that $\ulcorner(\mathbf{T} \downarrow \mathbf{T}) \downarrow (\mathbf{U} \downarrow \mathbf{U})\urcorner$ expresses the same truth-function as $\ulcorner\mathbf{R}\&\mathbf{S}\urcorner$. Here's a sub-proof of my claim:

   (i) A truth-value assignment makes $\ulcorner\mathbf{R}\&\mathbf{S}\urcorner$ true iff it makes both **R** and **S** true (by defintion of '&').

   (ii) A truth-value assignment makes both **R** and **S** true iff it makes both **T** and **U** true and it (by 2).

   (iii) A truth-value assignment makes both **T** and **U** true iff it makes $\ulcorner(\mathbf{T} \downarrow \mathbf{T}) \downarrow (\mathbf{U} \downarrow \mathbf{U})\urcorner$ true (by the definition of '↓'. I leave the verification of this as an exercise for the reader).

   (iv) So a truth-value assignment makes $\ulcorner\mathbf{R}\&\mathbf{S}\urcorner$ true iff it makes $\ulcorner(\mathbf{T} \downarrow \mathbf{T}) \downarrow (\mathbf{U} \downarrow \mathbf{U})\urcorner$ true.

   (v) So $\ulcorner\mathbf{R}\&\mathbf{S}\urcorner$ expresses the same truth-function as $\ulcorner(\mathbf{T} \downarrow \mathbf{T}) \downarrow (\mathbf{U} \downarrow \mathbf{U})\urcorner$.

4. $\ulcorner(\mathbf{T} \downarrow \mathbf{T}) \downarrow (\mathbf{U} \downarrow \mathbf{U})\urcorner \in L_2$ (by the definition of L_2).

5. So there exists a sentence in L_2 that expresses the same truth-function as $\ulcorner\mathbf{R}\&\mathbf{S}\urcorner$.

6. So, assuming the antecedent of Inductive Step, for every sentence of the form $\ulcorner\mathbf{R}\&\mathbf{S}\urcorner$, where $\mathbf{R}, \mathbf{S}$ have n or fewer connectives, there is a sentence in L_2 that expresses the same truth-function as $\ulcorner\mathbf{R}\&\mathbf{S}\urcorner$.

That proves Inductive Step for case (b).

Case (c):

1. Suppose that, for every sentence $\mathbf{P} \in L_1$ containing n or fewer connectives there is a sentence in L_2 that expresses the same truth-function as **P** (i.e., suppose the antecedent of Inductive Step).

2. Then there is a sentence in L_2 that expresses the same truth-function as **R** and a sentence of L_2 that expresses the same truth-function as **S** (as both **R** and **S** contain n or fewer connectives). Call those sentence in L_2 **T** and **U**, respectively.

3. I claim that $\ulcorner(\mathbf{T} \downarrow \mathbf{U}) \downarrow (\mathbf{T} \downarrow \mathbf{U})\urcorner$ expresses the same truth-function as $\ulcorner\mathbf{R} \vee \mathbf{S}\urcorner$. Here's a sub-proof of my claim:

   (i) A truth-value assignment makes $\ulcorner\mathbf{R} \vee \mathbf{S}\urcorner$ true iff it does not make both **R** and **S** false (by defintion of '∨').

   (ii) A truth-value assignment does not make both **R** and **S** false iff it does not make both **T** and **U** false and it (by 2).

   (iii) A truth-value assignment does not makes both **T** and **U** false iff it makes $\ulcorner(\mathbf{T} \downarrow \mathbf{U}) \downarrow (\mathbf{T} \downarrow \mathbf{U})\urcorner$ true (by the definition of '↓'. I leave the verification of this as an exercise for the reader).

(iv) So a truth-value assignment makes $\ulcorner \mathbf{R} \vee \mathbf{S} \urcorner$ true iff it makes $\ulcorner (\mathbf{T} \downarrow \mathbf{U}) \downarrow (\mathbf{T} \downarrow \mathbf{U}) \urcorner$ true.

(v) So $\ulcorner \mathbf{R} \vee \mathbf{S} \urcorner$ expresses the same truth-function as $\ulcorner (\mathbf{T} \downarrow \mathbf{U}) \downarrow (\mathbf{T} \downarrow \mathbf{U}) \urcorner$.

4. $\ulcorner (\mathbf{T} \downarrow \mathbf{U}) \downarrow (\mathbf{T} \downarrow \mathbf{U}) \urcorner \in L_2$ (by the definition of L_2).

5. So there exists a sentence in L_2 that expresses the same truth-function as $\ulcorner \mathbf{R} \& \mathbf{S} \urcorner$.

6. So, assuming the antecedent of Inductive Step, for every sentence of the form $\ulcorner \mathbf{R} \vee \mathbf{S} \urcorner$, where $\mathbf{R}, \mathbf{S}$ have n or fewer connectives, there is a sentence in L_2 that expresses the same truth-function as $\ulcorner \mathbf{R} \vee \mathbf{S} \urcorner$.

That proves Inductive Step for case (c).

And that finishes the proof of Inductive Step

And that finishes the proof that for every sentence $\mathbf{P} \in L_1$, there is a sentence in L_2 that expresses the same truth function as $\mathbf{P}$.

And that finishes the proof that {'↓'} is TF-complete.

Q.E.D.

Question 3

There are, in fact, two ways to use the proof that SD is sound to prove that SD* is sound. One way is to show that if $\Gamma \vdash \mathbf{P}$ in SD*, then $\Gamma \vdash \mathbf{P}$ in SD, so, by the soundness of SD, if $\Gamma \vdash \mathbf{P}$ in SD*, then $\Gamma \models \mathbf{P}$ — i.e., SD* is sound.

I'm going to do it the other way, which is the way I think the question prompt is nudging you. The other way goes as follows.

The goal is to prove that SD* is sound; i.e., that...

SD* Soundness: If $\Gamma \vdash \mathbf{P}$ in SD*, then $\Gamma \models \mathbf{P}$

...is true. We are going to prove it by mathematical induction. Let $\mathbf{P_k}$ be the $\mathbf{k}$th sentence in an SD* derivation, and let $\Gamma_{\mathbf{k}}$ be the set of assumptions open on the $\mathbf{k}$th line of that SD* derivation. Clearly, SD* Soundness follows from the following:

Basis Clause: $\Gamma_1 \models \mathbf{P}_1$.

Inductive Step: If $\Gamma_{\mathbf{k}+1} \vdash \mathbf{P}_{\mathbf{k}+1}$ in SD*, then $\Gamma_{\mathbf{k}+1} \models \mathbf{P}_{\mathbf{k}+1}$.

The proof of Base Case is exactly the same as the proof for the analogous base case in TLB on page 260, which is part of the proof of the soundness of SD. The proof of Inductive Step is exactly the same as the proof of the analogous inductive step on pages 260-264 of TLB, which is also part of the proof of soundness of SD, except you need to add a case for Backwards Conditional Introduction.

One way for that case to go is as follows:

Case 13: $\mathbf{P_{k+1}}$ is justified by Backwards Conditional Introduction:

$$\begin{array}{l|l}
\mathbf{h} & \quad \sim \mathbf{R} \\
\mathbf{j} & \quad \sim \mathbf{Q} \\
\mathbf{k}+1 & \mathbf{R} \supset \mathbf{Q} \quad (= \mathbf{P_{k+1}}) \qquad \mathbf{h}\text{-}\mathbf{j}, \text{B}\supset\text{I}
\end{array}$$

1. By exactly the same proof as in case 8, we know that $\Gamma_{\mathbf{k+1}} \models \ulcorner \sim \mathbf{R} \supset \sim \mathbf{Q} \urcorner$.
2. So there is no truth-value assignment such that every member of $\Gamma_{\mathbf{k+1}}$ is true and $\ulcorner \sim \mathbf{R} \supset \sim \mathbf{Q} \urcorner$ is false (by the definition of '$\models$').
3. So there is no truth-value assignment such that every member of $\Gamma_{\mathbf{k+1}}$ is true and $\ulcorner \sim \mathbf{R} \urcorner$ is true and $\ulcorner \sim \mathbf{Q} \urcorner$ is false (by the definition of '$\supset$').
4. So there is no truth-value assignment such that every member of $\Gamma_{\mathbf{k+1}}$ is true and $\mathbf{R}$ is false and $\mathbf{Q}$ is true (by the definition of '$\sim$').
5. So there is no truth-value assignment such that every member of $\Gamma_{\mathbf{k+1}}$ is true and $\mathbf{Q} \supset \mathbf{R}$ is false (again by the definition of '$\supset$').
6. So $\Gamma_{\mathbf{k+1}} \models \mathbf{Q} \supset \mathbf{R}$.

So Inductive Step is true in the case when $\mathbf{P_{k+1}}$ is justified by Backwards Conditional Introduction. So, by that and the stuff in the proof of the soudness of SD, Inductive Step is true. So, by that and Base Case, SD* is sound.

Q.E.D.

24.241 Logic I
Fall 2009