Logic I – Session 9

Plan

- One more example to practice meta-language proofs

- Explain mathematical induction

- Examples

More practice

3.6E (2b)   If $\Gamma \cup \{P\} \vDash Q$, then $\Gamma \vDash P \supset Q$

1. $\Gamma \cup \{P\} \vDash Q$
2. Every TVA that makes every member of $\Gamma \cup \{P\}$ true also makes Q true
3. Every TVA that makes every member of Γ true and P true also makes Q true.
4. No TVA makes every member of Γ true and P true but makes Q false.
5. So no TVA that makes every member of Γ true makes $P \supset Q$ false.
6. So every TVA that makes every member of Γ true also makes $P \supset Q$ true.
7. So $\Gamma \vDash P \supset Q$

Remember:

- Truth-functional notions are all defined in terms of TVAs (not derivability or possibility)

- E.g.: Γ is truth-functionally inconsistent iff there's no TVA that makes every member of Γ true.

- $\vDash$ vs. $\vdash$ (The symbol for derivability looks like a piece of a derivation.)

Mathematical induction

- If we have an ordered sequence of infinitely many things, we can prove that all of them have a property F by proving two simpler claims:

  - **Basis clause**: Prove that the first thing in the sequence is F

  - **Inductive step:** Prove that if the nth thing is F, then so is the n+1st.

  - Conclude that everything in the sequence is F

- The things in the sequence can be anything, e.g. sentences of SL or sets of sentences SL

First Example of MI

- 6.1E (1a) Prove: No sentence of SL that contains only binary connectives, if any, is truth-functionally false.

- First step: insert relevant definitions in the thesis to be proven

- Prove: No sentence of SL that contains only binary connectives is false on every truth-value assignment

- Equivalently:

- Prove: Every sentence of SL that contains only binary connectives is true on some TVA

- Second step: Arrange our sentences into a sequence. We can do that by using the number of connectives in the sentences. (This will be a common strategy)

- Third step: Find our basis clause and inductive clause. We sometimes get them by directly applying our thesis to the members of our sequence.

- So we might start with:

  - Basis clause: Every sentence with zero occurrences of a binary connective (and no occurrence of `~') is true on some TVA.

  - Inductive step: if (A) every sentence with n occurrences of binary connectives or fewer (and no occurrence of `~') is true on some TVA, then (C) every sentence with n+1 occurrences of binary connectives (and no occurrence of `~') is true on some TVA.

- But sometimes we need to look ahead and think about why, intuitively, the thesis we're proving is true. Doing that might help us see that it's easier to prove something stronger!

- In this case we can see that if we have a TVA that makes atomic sentences **P** and **Q** true, it will also make these true: (**P**&**Q**), (**P**v**Q**), (**P**⊃**Q**), (**P**≡**Q**).

- That is, if we have a TVA that makes all atomic sentences true, we can never get a false compound with negation!

- So we should be able to use this TVA to prove the thesis that every sentence with only binary connectives is true on some TVA.

- So instead of having as our basis clause: Every sentence with zero occurrences of a binary connective (and no occurrence of `~`) is true on **some** TVA...

- ...we might want to work with the **particular** TVA that looks like it will work and then infer the more general claim later. So:

- Basis clause: Every sentence with zero occurrences of a binary connective (and no occurrence of `~`) is true on the TVA that makes every atomic sentence true.

Fourth step: Prove the basis clause.

This is trivial. Every sentence with zero occurrences of a binary connective (and no occurrence of negation) is atomic.

Every atomic sentence is true on the TVA on which every atomic sentence is true.

So every sentence with zero occurrences of a binary connective (and no occurrence of negation) is true on the TVA on which every atomic sentence is true.

- Fifth step: Prove the inductive step.

- That is, prove that if (A), then (C).

  - (A) Every sentence with n or fewer occurrences of binary connectives (and no occurrence of `~') is true on the TVA on which every atomic sentence is true, then

  - (C) Every sentence with n+1 occurrences of binary connectives (and no occurrence of `~') is true on the TVA on which every atomic sentence is true.

- So assume (A).

- (A) Every sentence with n occurrences or fewer of binary connectives (and no occurrence of `~') is true on the TVA on which every atomic sentence is true.

- Now, if a sentence **R** has n+1 binary connectives, then it is of one of the forms (**P**&**Q**), (**P**v**Q**), (**P**⊃**Q**), (**P**≡**Q**).

- Furthermore, **P** and **Q** must have fewer binary connectives than **R** does, that is, fewer than n+1.

- That means, by assumption (A), that **P** and **Q** are both true on the TVA that makes every atomic sentence true.

- So we've gotten that **P** and **Q** are both true on the TVA that makes every atomic sentence true.

- Now, whatever the main connective of **R** is, **R** is true whenever its immediate components are true. That is, whenever **P** and **Q** are true.

- So **R** is true on the TVA that makes every atomic sentence true.

- So we've shown that if (A) is true, then any sentence **R** with n +1 occurrences of binary connective is true on the TVA that makes every atomic sentence true.

- That is, we've shown that if (A), then (C).

- So we've proven the inductive step, and the proof is complete!

Second Example of MI

- Let SL- be a language just like SL except that it contains only binary connectives

- Prove: For every sentence **P** of SL-, **P** does not contain two connectives without at least one parenthesis in between

- To prove this by MI, we need to arrange the sentences of SL- into some kind of order

- We can use the # of binary connectives in a sentence to do it

- Basis clause: For every sentence **P** of SL- with zero connectives, **P** does not contain two connectives without at least one parenthesis in between

- Proof: This is easy. An SL- sentence with zero connectives certainly doesn't contain two connectives. So it does not contain two connectives without at least one parenthesis in between.

- Next we have to prove the inductive step:

- Prove that if (A) then (C)

- (A) For every SL- sentence **P** with n connectives or fewer, **P** does not contain two connectives without at least one parenthesis in between

- (C) For every SL- sentence **Q** with n+1 connectives, **Q** does not contain two connectives without at least one parenthesis in between

- (A) For every SL- sentence **P** with n connectives or fewer, **P** does not contain two connectives without at least one parenthesis in between

- Now suppose **Q** has n+1 connectives

- There are four cases for the form **Q** could have: (**R&S**), (**RvS**), (**R⊃S**), (**R≡S**). It cannot be atomic, since n+1>0.

- Our argument will not depend on what the main connective is, so let us represent it as `?'. Then **Q** is of the form (**R ? S**).

- Now, since **Q** has n+1 connectives, **R** and **S** must each have n or fewer connectives. So, (A) applies to **R** and **S**: neither contains two connectives without a parenthesis in between.

- Now, the only other connective in **Q** is the main connective we are representing with `?'. So the only way **Q** could contain two connectives without a parenthesis in between, is if there were no parenthesis between ? and some connective in **R** or **S**.

- But since **R** and **S** are sentences of SL-, they are either atomic or have parentheses around the outside.

- If **R** and **S** are both atomic, then **Q** contains only 1 connective, and hence does not contain two connectives without at least one parenthesis in between.

- If on the contrary at least one of **R** and **S**, say **R,** is molecular, then it has parentheses around the outside.

  - But since **R** is adjacent to ?, an outer parenthesis of **R** is between ? and any other symbol in **R**. The same goes for **S** if it is molecular.

- So either way, there is no connective anywhere in **R** or **S** without a parenthesis between it and ?.

- Since the only way there could have been two connectives in **Q** without at least one parenthesis in between was if ? was one of them, there are no two connectives anywhere in **Q** without at least one parenthesis in between.

- So since **Q** has n+1 connectives, (C) is true. So if (A) then (C). So the inductive step is proved, and the argument complete.

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