Logic I - Session 22

Meta-theory for predicate logic

The course so far

- Syntax and semantics of SL
- English / SL translations
- TT tests for semantic properties of SL sentences
- Derivations in SD
- Meta-theory: SD is adequate for SL (sound, complete)

- Syntax and semantics of PL
- English / PL translations
- Derivations in PD
- Next: PD is adequate for PL (sound, complete)

Soundness, Completeness

- There are meta-theoretical results for PD as well as PDE. In particular:
  - If Γ is a set of PL sentences and **P** is a PL sentence, then $\Gamma \vDash$ **P** iff $\Gamma \vdash$ **P** in PD.
  - If Γ is a set of PLE sentences and **P** is a PLE sentence, then $\Gamma \vDash$ **P** iff $\Gamma \vdash$ **P** in PDE.
- We'll focus on PL and PD, coming back to PLE and PDE later if we have time.

Soundness

- We'll focus on soundness today.
  - If $\Gamma \vdash \mathbf{P}$ in PD, then $\Gamma \vDash \mathbf{P}$.
  - To prove: If there's a PD derivation all of whose primary assumptions are members of Γ and in which **P** occurs only in the scope of those assumptions, then **P** is quantificationally entailed by Γ.

Soundness

- As with soundness for SD, we prove our result by proving something stronger:
  - Every sentence in a PD derivation is q-entailed by the set of assumptions with scope over it.
- Our proof of this will appeal to a mathematical induction analogous to the one we used to prove the soundness of SD.

Soundness

- Let Γi be the set of assumptions open at line i in a derivation, and let **P**i be the sentence on line i.
- Basis clause: $\Gamma 1 \vDash$ **P**1.
- Inductive step: If $\Gamma i \vDash$ **P**i for all $i \leq k$, then $\Gamma k+1 \vDash$ **P**k+1.
  - We'll prove this by cases, one case for each rule that could have justified line k+1.
- Conclusion: For every line k in a derivation, $\Gamma k \vDash$ **P**k.
  - I.e.: Every sentence in a PD derivation is q-entailed by the set of assumptions with scope over it.

Soundness: Basis clause

- To prove: Γ1 ⊨ **P**1.
  - = No interpretation mem Γ1 true but makes **P**1 false.
- The first line of any derivation is an assumption.
- Every assumption counts as being in its own scope, so **P**1 is in the scope of **P**1 and only **P**1.
  - I.e. Γ1 = {**P**1}.
- Trivially, any interpretation that makes **P**1 true makes **P**1 true. So {**P**1} ⊨ **P**1. So Γ1 ⊨ **P**1.

Soundness: Inductive step

- Suppose $\Gamma_i \vDash \mathbf{P}_i$ for all lines $i \leq k$ in a derivation.
  - To prove: $\Gamma_{k+1} \vDash \mathbf{P}_{k+1}$.
  - Strategy: Line k+1 must be justified by some rule, and no matter what rule it is, we have $\Gamma_{k+1} \vDash \mathbf{P}_{k+1}$.
- We have all 12 rules from SD, so we need the result to hold for those cases.
  - Our proofs don't need much adjustment...

Inductive step: &I

- Suppose line k+1 is justified by &I.
- Then **P**k+1 is justified by two earlier lines i and j, and **P**k+1 is of the form **Q**i & **R**j.
- So we should prove: Γk+1 ⊨ **Q**i & **R**j
  - It suffices to prove that Γk+1 ⊨ **Q**i and Γk+1 ⊨ **R**j.
  - Why? Fill in defs of '⊨' and '&'.

Inductive step: &I

- If any I that mem $\Gamma k+1$ true makes **Q**i true, and
  any I that mem $\Gamma k+1$ true makes **R**j true, then:
  any I that mem $\Gamma k+1$ true makes **Q**i true and **R**j true.
- If I makes **Q**i is true and **R**j true, then I makes **Q**i & **R**j true. [By our semantics for '&']
- If any I that mem $\Gamma k+1$ true makes **Q**i true, and
  any I that mem $\Gamma k+1$ true makes **R**j true, then:
  any I that mem $\Gamma k+1$ true makes **Q**i & **R**j true.
- I.e.: if $\Gamma k+1 \vDash$ **Q**i and $\Gamma k+1 \vDash$ **R**j, then $\Gamma k+1 \vDash$ **Q**i & **R**j.

Inductive step: &I

- OK, so prove: $\Gamma k+1 \vDash \mathbf{Q}i$ and $\Gamma k+1 \vDash \mathbf{R}j$.
- $\Gamma i \vDash \mathbf{Q}i$ and $\Gamma j \vDash \mathbf{R}j$ [by the inductive hypothesis, since i and j are earlier than k+1]
- $\Gamma i \subseteq \Gamma k+1$ and $\Gamma j \subseteq \Gamma k+1$
- So we want:
  - If $\Gamma i \vDash \mathbf{Q}i$ and $\Gamma i \subseteq \Gamma k+1$, then $\Gamma k+1 \vDash \mathbf{Q}i$, and if $\Gamma j \vDash \mathbf{R}j$, and $\Gamma j \subseteq \Gamma k+1$ then $\Gamma k+1 \vDash \mathbf{R}j$.
- So we just need 11.3.2, which is easy:
  - If $\Gamma \vDash \mathbf{P}$ and $\Gamma \subseteq \Gamma^*$, then $\Gamma^* \vDash \mathbf{P}$.

Inductive step: &I

- So now we know: $\Gamma k{+}1 \vDash \mathbf{Q}i$ and $\Gamma k{+}1 \vDash \mathbf{R}j$.
- So from this, we know that if line k+1 is justified by &I, then $\Gamma k{+}1 \vDash \mathbf{Q}i \ \& \ \mathbf{R}j$.
- So the first case of our proof of the inductive step is complete. We need to prove analogous results for all other rules of PD.
- The interesting ones are the new ones, the rules for quantifiers.

Inductive step: ∀E

- Suppose **P**k+1 is justified by ∀E.
- Then **P**k+1 is of the form **Q**(a/x) and is justified by applying ∀E to some earlier line i containing (∀x)**Q**.
- Γi ⊨ (∀x)**Q**   [by inductive hypothesis]
- Γi ⊆ Γk+1
- So Γk+1 ⊨ (∀x)**Q**   [again, by 11.3.2]
- So we just need to show that:
  if Γk+1 ⊨ (∀x)**Q**, then Γk+1 ⊨ **Q**(a/x).
  - This will follow if {(∀x)**Q**} ⊨ **Q**(a/x).

Inductive step: $\forall$E

- We'll sketch a proof of the general claim 11.1.4:
  - For any **a**, **x**, **Q**, {(∀**x**)**Q**} ⊨ **Q(a/x)**.
- Suppose I makes **(**∀**x)Q** true. Then for every d for I:
  - d satisfies **(**∀**x)Q**.
  - So for every u∈UD, d[u/**x**] satisfies **Q**. [By df. sat.]
  - So for any **a**, d[I(**a**)/**x**] satisfies **Q**.
  - d[I(**a**)/**x**] satisfies **Q** iff d satisfies **Q(a/x)** [11.1.1]
  - So d satisfies **Q(a/x)**.
- So any I that makes **(**∀**x)Q** true makes **Q(a/x)** true.

Inductive step: ∀E

- If any I that makes **(∀x)Q** true makes **Q(a/x)** true, that just means that {**(∀x)Q**} ⊨ **Q(a/x)**.
- So now we know that:

  if Γk+1 ⊨ (∀x)**Q**, then Γk+1 ⊨ **Q**(a/x).

- And we knew that Γk+1 ⊨ (∀x)**Q**.
- So Γk+1 ⊨ **Q**(a/x).
- **Q**(a/x) was just **P**k+1, so we've shown that if **P**k+1 is justified by ∀E, then Γk+1 ⊨ **P**k+1.

Progress report

- Recall, we're trying to prove PD's soundness by proving that no matter what rule you use to get **P**k+1, it's q-entailed by Γk+1.
- We've talked about the SD rules and ∀E. To complete our proof of PD's soundness, we need to check the remaining quantifier rules.
- Let's do one more. The most complicated is ∃E.

Inductive step: ∃E

- Suppose $\mathbf{P}_{k+1}$ is justified by ∃E.
  - Then we have lines:

h	$(\exists x)Q$	
j	| $Q(a/x)$	
m	| P_{k+1}	
k+1	P_{k+1}	∃E h, j–m

  - And **a** can't be in $\mathbf{P}_{k+1}$, $(\exists \mathbf{x})\mathbf{Q}$, or in any open assumptions
- Note: $\Gamma_m = \Gamma_h \cup \{Q(a/x)\}$.

Inductive step: ∃E

- $\Gamma m \vDash \mathbf{P}k{+}1$ [by inductive hypothesis]
- So $\Gamma h \cup \{\mathbf{Q(a/x)}\} \vDash \mathbf{P}k{+}1$ [since $\Gamma m = \Gamma h \cup \{\mathbf{Q(a/x)}\}$]
- And $\Gamma h \cup \{\mathbf{Q(a/x)}\} \subseteq \Gamma k{+}1 \cup \{\mathbf{Q(a/x)}\}$ [since $\Gamma h \subseteq \Gamma k{+}1$]
- So $\Gamma k{+}1 \cup \{\mathbf{Q(a/x)}\} \vDash \mathbf{P}k{+}1$ [by 11.3.2]
- Also, $\Gamma k{+}1 \vDash \mathbf{(\exists x)Q}$ [since $\Gamma h \subseteq \Gamma k{+}1$ and $\Gamma h \vDash \mathbf{(\exists x)Q}$]
- So what we need is this:
  - If $\Gamma k{+}1 \vDash \mathbf{(\exists x)Q}$ and $\Gamma k{+}1 \cup \{\mathbf{Q(a/x)}\} \vDash \mathbf{P}k{+}1$, then $\Gamma k{+}1 \vDash \mathbf{P}k{+}1$ (assuming our restrictions on **a**)

Inductive step: ∃E

- So, assume (i) **a** doesn't occur in **P**k+1, **(∃x)Q**, or in any member of Γk+1, (ii) Γk+1 ⊨ **(∃x)Q**, and (iii) Γk+1 ∪ {**Q(a/x)**} ⊨ **P**k+1.
  - Prove Γk+1 ⊨ **P**k+1 by reductio.
- Assume some I mem Γk+1 true and makes **P**k+1 false.
- Then by (ii), I makes **(∃x)Q** true.
- So for any d for I, d satisfies **(∃x)Q**. [by def. truth]
- For for any d, there's some u∈UD s.t. d[u/**x**] satisfies **Q** on I [by def. satisf.]

Inductive step: ∃E

- Let I′ be just like I except that I′(**a**)=u.
- What object **a** denotes is irrelevant to whether an interpretation makes sentences without **a** true.
- So I′ still mem Γk+1 true and **P**k+1 false
- Since I′ mem Γk+1 true, by (ii) it makes **(∃x)Q** true

Inductive step: ∃E

- And I′ makes **Q(a/x)** true. Because...
  - d[**u/x**] satisfies **Q** on I′ (I′ only differs from I on **a**)
  - Since I′(**a**)=u, d[u/x] satisfies **Q(a/x)** on I′ too
  - In fact, **Q(a/x)** has no free variables, no **x**, in it
  - So d itself satisfies **Q(a/x)** on I′

    (taking an **x**-variant of d isn't necessary)
  - A <u>sentence</u> is satisfied by every assignment on an I if it's satisfied by some assignment on I.
  - So every assignment satisfies **Q(a/x)** on I′
  - So by def., **Q(a/x)** is true on I′

Inductive step: $\exists$E

- So we have:
  - Γk+1's members are all true on I′
  - **P**k+1 is false on I′
  - **($\exists$x)Q** is true on I′
  - **Q(a/x)** is true on I′
- Recall assumption (iii): Γk+1 $\cup$ {**Q(a/x)**} $\vDash$ **P**k+1.
- So **P**k+1 is true on I′.
- Contradiction.
- So not: some I mem Γk+1 true and makes **P**k+1 false.

Inductive step: ∃E

- Recall, we showed earlier that what we need is this:
  - If $\Gamma k{+}1 \vDash$ **(∃x)Q** and $\Gamma k{+}1 \cup$ {**Q(a/x)**} $\vDash$ **P**k+1, then $\Gamma k{+}1 \vDash$ **P**k+1 (assuming our restrictions on **a**)
- We assumed (i) **a** doesn't occur in **P**k+1, **(∃x)Q**, or in any member of $\Gamma k{+}1$, (ii) $\Gamma k{+}1 \vDash$ **(∃x)Q,** and (iii) $\Gamma k{+}1 \cup$ {**Q(a/x)**} $\vDash$ **P**k+1.
  - And we showed this implied $\Gamma k{+}1 \vDash$ **P**k+1.
- So we've shown that line k+1 is q-entailed by the open assumptions with scope over it when it's justified by ∃E