Philosophy 244: #9— Modal Predicate Logic

Now we turn to modal predicate logic, the result of adding modal operators to first-order quantification theory. First-order quantification theory is itself the result of making two additions to propositional logic. *First*, we dig into the structure of atomic sentences, replacing the simple indivisible *p* with things of the form

$\varphi x_1 x_2 .... x_k$

that is, a k-ary predicate followed by k variables. *Second*, we introduce a new way of forming complex sentences from simpler ones, a way that wouldn't have been available without the aforementioned digging. Whenever you've got a formula $\alpha(x)$ with a variable in it, you can stick a quantifier in front.

$\forall x \alpha(x)$
$\exists x \alpha(x)$

If flexibility is the property that $\varphi(x)$ attributes to *x*, then the first of these says that everything is flexible, and the second that something is flexible. The process can be re-iterated with different variables, to get things like $\forall x \exists y \ \psi(x, y)$, which if admiration is the relation expressed by ψ, says that everyone admires somebody. And so on indefinitely.

Syntax

Now let's give the syntax and semantics formally; that we'll be taking the universal quantifier $\forall$ as basic, the existential $\exists$ will be defined as $\neg\forall\neg$. A language $\mathcal{L}$ for LPC has as its lexicon

The book's name for first-order quantification theory is "lower predicate calculus" or LPC. The reason for the "lower" is that we're quantifying into name-position only, not predicate-positions as in second-order logic ($\exists X$ *Xmc = Marcus and Kripke have something in common*). "Lower" means the same as "first-order."

for each $n \geq 1$, a denumerably infinite set of *n*-place predicates $\varphi. \ \psi,...(P, Q,,.....)$
a denumerably infinite set of individual variables *x, y, z,...*
the three logical symbols $\neg, \vee, \forall$
left and right parentheses

The formation rules for wffs are

FR1 An *n*-place predicate followed by *n* individual variables is an atomic wff
FR2 If α is a wff so is $\neg\alpha$
FR3 If α, β are wffs, so is $(\alpha \vee \beta)$
FR4 If α is a wff and *x* a variable then $\forall x \ \alpha$ is a wff.

The definitions of &, $\supset$, and $\equiv$ are as before. The existential quantifier is defined by

D∃ $\exists x \ \alpha$ =df $\neg\forall x \neg \alpha$

The scope of a quantifier in α is the smallest sub-wff of α that contains it. (Examples.) A variable *x* is called bound or free according to whether it is or isn't in the scope of an *x*-quantifier. Note that it is really occurrences of variables that are bound or free, and that bound/free is always relative to an enclosing formula; *x* is bound in $\exists x \forall y Rxy$ but free in the bit after the initial quantifier: $\forall y \ Rxy$.

An exception is made though for the variable immediately after the $\forall$ or $\exists$; this is neither bound nor free, it's considered just part of the quantifier.

Semantics

The first thing we need for an interpretation of $\mathcal{L}$ is a *domain* D to signal what we are talking about. When we say $\exists x\alpha$ or $\forall x\alpha$, this means that there is something in the domain which satisfies α, or that everything in the domain satisfies α. The domain can in principle be considered our interpretation of the quantifiers—D = V($\forall$)—in something like the way that multiplication is our interpretation of conjunction; in that case the model would be determined by V alone. If P is an *n*-place predicate, V(P) is a set of *n*-tuples drawn from the domain D. In the language of cross products, it's a subset of D×...×D (*n* times). A model $\mathcal{M}$ of $\mathcal{L}$ is an ordered pair <D,V> of a domain and an interpretation function.

Are the cases really analogous? What does this mean for $\forall$'s status as a logical constant?

Don't variables have to stand for objects? Yes and no. Names when we get to them *will* be assigned objects by V. But not variables; they stand for "arbitrary" or "unspecified" objects. The reason is that there are two kinds of wffs. Those with free variables — *open* wffs – we don't assign truth-values to, so there's no need to specify an object to help determine that truth-value. Those whose variables are all bound— *closed* wffs or *sentences* of $\mathcal{L}$ — we do assign truth-values to, but in a way that relies only on the *pattern* of variables in the sentence, and so does not require an assignment of particular objects to any of them.

Kit Fine has a book taking arbitrary objects seriously as a distinctive sort of object An arbitrary D has the properties all Ds have in common, with a few exceptions, e.g., non-arbitrariness.

Consider $\forall xPx$. This will be true if V(P) = D, otherwise false. No need for x to stand for any particular entity. As for Px, whether it should be counted true or false depends on what we think of x as standing for. We *could* let V assign values to variables too, but then we lose this freedom of construing it as standing for whatever we like. Also it is mainly sentences we want to come out true or false in the model, so assigning a fixed value to x would not accomplish much.

Open wffs will be evaluable only if their universal closures are valid or anti-valid.

But again, though we don't don't want x to refer to anything in particular in our model <D,V>, we *do* want to generalize over all the things it might be taken as referring to. This in order to say, e.g., that $\forall xPx$ is true iff Px is true "whatever we take x to be." To accomplish this, let a value-assignment μ be a function mapping variables to arbitrary members of the domain; for all variables x, $\mu(x)\epsilon$D. We write $V_\mu(\alpha)$=1 to mean that α is true in the model <D,V> when the variables are given the values assigned by μ. This approach enabled in 1930 the first recursive definition of truth for quantified languages.

Tarski called μ a "sequence," and spoke of satisfaction *by* a sequence of objects rather than truth *of* those objects.

(Vφ) $V_\mu(\varphi(x_1....x_n))$=1 iff $<\mu(x_1)...\mu(x_n)>\epsilon$V($\varphi$), otherwise 0.

(V¬) $V_\mu(\neg\alpha)$ = 1 iff $V_\mu(\alpha)$=0, otherwise 0

(V∨) $V_\mu(\alpha\vee\beta)$ = 1 iff $V_\mu(\alpha)$=1 or $V_\mu(\beta)$=1

(V∀) $V_\mu(\forall x\alpha)$=1 iff $V_\rho(\alpha)$ = 1 for every x-alternative ρ to μ, otherwise 0

The interesting part of the definition, and the part we most need the value-assignment μ for, comes with the quantifiers. Say that ρ is an x-alternative to μ if ρ agrees with μ on every variable but (possibly) x.

From this last and the rule for negation it follows that

(V∃) $V_\mu(\exists x\alpha)$=1 iff $V_\rho(\alpha)$=1 for some x-alternative ρ to μ, otherwise 0.

Finally the all important notion of

Validity α is valid in $\mathcal{M}$ = <D, V> iff $V_\mu(\alpha)$=1 for every variable-assignment μ, and valid (period) iff it's valid in every model $\mathcal{M}$.

Axiomatization

This time we'll state the axiom(s) schematically instead of leaning hard on a a uniform substitution rule.

PC Any LPC substitution-instance of a valid PC wff is an axiom of LPC.

$\forall 1$ $\forall x\alpha \supset \alpha[y/x]$

The precise definition of $\alpha[y/x]$ is complicated, because we don't want x's free in α to turn into y's that are bound in $\alpha[y/x]$. See *The Principle of Replacement* on pp.240-241.

The rules are

MP ⊢α, ⊢α⊃β ⇒ ⊢ β

∀2 ⊢(α⊃β) ⇒ ⊢(α⊃∀xβ) provided *x* is not free in α.

Some derived rules:

UG ⊢α ⇒ ⊢∀xα
UG⊃ ⊢(α⊃β) ⇒ ⊢(∀xα⊃∀xβ)
UG≡ ⊢(α≡β) ⇒ ⊢(∀xα≡∀xβ)

Eq ⊢α≡β ⇒ ⊢γ[α]≡γ[β]
    where γ[α] differs from γ[β] only in having α at 0 or more places where γ[β] has β

RBV ⊢∀xα≡∀yβ
    where α differs from β only in having free *x* where and only where β has free *y*

Some theorems:

  LPC1 ∀x(α⊃β)⊃(∀xα⊃∀xβ)

  LPC2 ∀x(α⊃β)⊃(α ⊃ ∀xβ) provided *x* is not free in α

  LPC3 ∃y(α[y/x]⊃∀xα) if *y* is not free in ∀xα
  QI ¬∃x¬α≡∀xα — quantifier interchange, generalizes a la LMI

Modal LPC

The language of modal LPC differs from the language of LPC in having the modal operator □ as a lexical item, and in having a slightly amended second formation rule:

FR2' If α is a wff then so are ¬α and □α

    A model $\mathcal{M}$ for modal LPC is a quadruple <W,R,D,V> in which <W,R> is a frame, D is a domain, and V is a function from n-place predicates to, not *n*-tuples of domain elements, but *n+1*-tuples of domain elements and worlds — the idea being that <a,b,*w*>εV(φ) iff a and b stand in the relation expressed by φ in world *w*.

  (Vφ) $V_\mu(\varphi(x_1....x_n),w)$=1 iff $< \mu(x_1)...\mu(x_n), w >\varepsilon V(\varphi)$, otherwise 0.

  (V¬) $V_\mu(\neg\alpha,w)$ = 1 iff $V_\mu(\alpha,w)$=0, otherwise 0

  (V∨) $V_\mu(\alpha \vee \beta,w)$ = 1 iff $V_\mu(\alpha,w)$=1 or $V_\mu(\beta,w)$=1

  (V□) $V_\mu(\Box\alpha,w)$ = 1 iff $V_\mu(\Box\alpha,u)$ = 1 for every *u* that *w* bears R to

  (V∀) $V_\mu(\forall x\alpha,w)$=1 iff $V_\rho(\alpha,w)$=1 for all *x*-variants ρ of μ

Finally the all important notion of

  Validity α is valid in $\mathcal{M}$ = <WRDV> iff $V_\mu(\alpha,w)$=1 for all *w* and μ, and valid on <WR> iff
    it's valid in every model $\mathcal{M}$ based on <WR>.

Systems of Modal Predicate Logic

Suppose S is a normal system of modal propositional logic. Then LPC+S is defined as follows. Axioms:

S' ⊢α whenever α is an LPC substitution instance of an S-theorem,

∀1 ⊢∀xα⊃α[y/x] if α[y/x] is α with a free *y* replacing every free *x*

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Rules:

NE ⊢α ⇒ ⊢□α

MP ⊢(α⊃β), ⊢α ⇒ ⊢β

∀2 ⊢(α ⊃ β) ⇒ ⊢(α ⊃ ∀xβ) provided x is not free in α

Another important *potential* axiom is the following, which concerns interactions between the quantifiers and □:

BF ⊢ ∀x□α⊃□∀xα

This is the famous Barcan formula, named after Ruth Barcan Marcus. The notation S+BF is used for LPC+S with BF added. Let's see what the BF can do for us. The converse is important too.

BFC ⊢ □∀xα⊃∀x□α

How do these strike you intuitively? Everything is necessarily material, according to the materialist. Is it necessary that everything be material? Necessarily everything exists. Does it follow we are all like God in necessarily existing?

Recall LPC1 ∀x(α⊃β)⊃(∀xα⊃∀xβ). Can we generalize it to ∀x(α⊰β)⊃(∀xα⊰∀xβ)? Well, in box terms it's ∀x□(α ⊃β) ⊃ (□∀xα⊃□ ∀x β). To prove this, it's enough to show that ∀x□(α⊃β) ⊃ □∀x(α⊃ β) (since □ and ∀ distribute over material implication). But that's an instance of BF.

Which is more plausible, BF or CBF? The converse of the Barcan formula is, interestingly enough, a theorem even of LPC+K:

BFC   □∀xα⊃∀x□α
1   ∀xα ⊃ α   ∀1
2   □∀xα ⊃ □α   (1)xDR1
3   □∀xα ⊃ ∀x□α   (2)x∀2

How do modalities mix with the "opposite" sort of quantification? ∃x□α ⊃ □∃xα is valid, as is ◇∀xα ⊃ ∀x◇α. Converses of these are not valid. Neither would we want them to be. The converse of the first, □∃xα ⊃ ∃x□α, involves exactly the kind of mix-up that Quine complained about in quantified modal logic. An instance would be, *Necessarily there is a number of planets only if something necessarily numbers the planets*.

There are lots of modal systems S such that LPC+S does not have BF as a theorem. It is a theorem, though, if S contains the Brouwer axiom B. Here is why:

BF   ⊢ ∀x□α⊃□∀xα
1   ∀x□α ⊃ □α   ∀1
2   ◇∀x□α ⊃ ◇□α   (1)xDR3
3   ◇□α ⊃ α   B
4   ◇∀x□α ⊃ α   (2),(3)xPC
5   ◇∀x□α ⊃ ∀xα   (4)x∀2
6   ∀x□α ⊃ □∀xα   (5)xDR2

Next time: different systems of modal predicate logic, types of validity, soundness, essentialism, de re and de dicto necessity.

24.244 Modal Logic
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