Nov. 28 2005: Lecture 23:

Resonance Phenomena, Beam Theory

Reading:
Kreyszig Sections: §2.11 (pp:111–116) , §2.13 (pp:130–31)

Resonance Phenomena

The physics of an isolated damped linear harmonic oscillator follows from the behavior of the homogeneous equation:[14]

$$M\frac{d^2y(t)}{dt^2} + \eta l_o \frac{dy(t)}{dt} + K_s y(t) = 0 \tag{23-1}$$

The zero on the right-hand-side of Eq. 23-1 implies that there are no external forces applied to the system. The system oscillates with a characteristic frequency $\omega = \sqrt{K_s/M}$ with amplitude that are damped by a characteristic time $\tau = (2M)/(\eta l_o)$ (i.e., the amplitude is damped $\propto \exp(-t/\tau)$.)

Mathematica® Example: Lecture-23
Simulating Resonance by Iteration and Noise
Finite Differences with Noise and Forcing

[14] A concise and descriptive description of fairly general harmonic oscillator behavior appears at http://hypertextbook.com/chaos/41.shtml

A general model for a damped and forced harmonic oscillator is

$$M\frac{d^2y(t)}{dt^2} + \eta l_o \frac{dy(t)}{dt} + K_s y(t) = F_{app}(t) \tag{23-2}$$

where F_{app} represents a time-dependent applied force to the mass M.

General Solutions to Non-homogeneous ODEs . . . .

Equation 23-2 is a non-homogeneous ODE—the functions and its derivatives appear on one side and an arbitrary function appears on the other. The general solution to Eq. 23-2 will be the sum of two parts:

$$\begin{aligned} y_{gen}(t) &= y_{part}(t) + y_{homog}(t) \\ y_{gen}(t) &= y_{F_{app}}(t) + y_{homog}(t) \end{aligned} \tag{23-3}$$

$$y_{homg}(t) = \begin{cases} C_+ e^{-|\lambda+|t} + C_- e^{-|\lambda-|t} & (\eta l_o)^2 > 4MK_s \quad \text{Over-damped} \\ C_1 e^{-|\lambda|t} + C_2 t e^{-|\lambda|t} & (\eta l_o)^2 = 4MK_s \quad \text{Critical Damping} \\ C_+ e^{-|\mathrm{Re}\lambda|t} e^{\imath|\mathrm{Im}\lambda|t} + C_- e^{-|\mathrm{Re}\lambda|t} e^{-\imath|\mathrm{Im}\lambda|t} & (\eta l_o)^2 < 4MK_s \quad \text{Under-damped} \end{cases} \tag{23-4}$$

where $y_{part} \equiv y_{F_{app}}$ is the solution for the particular F_{app} on the right-hand-side and y_{homog} is the solution for the right-hand-side being zero. *Adding the homogeneous solution y_{homog} to the particular solution y_{part} is equivalent to adding a "zero" to the applied force F_{app}*

Interesting cases arise when the applied force is periodic $F_{app}(t) = F_{app}(t+T) = F_{app}(t + 2\pi/\omega_{app})$, especially when the applied frequency, ω_{app} is close to the the characteristic frequency of the oscillator $\omega_{char} = \sqrt{K_s/M}$.

Modal Analysis . . . .

For the case of a periodic forcing function, the time-dependent force can be represented by a Fourier Series. Because the second-order ODE (Eq. 23-2) is linear, the particular solutions for each term in a Fourier series can be summed. Therefore, particular solutions can be analyzed for one trigonometric term at a time:

$$M\frac{d^2y(t)}{dt^2} + \eta l_o \frac{dy(t)}{dt} + K_s y(t) = F_{app}\cos(\omega_{app} t) \tag{23-5}$$

There are three general cases for the particular solution:

	Condition	Solution for $F(t) = F_{app}\cos(\omega_{app}t)$
Undamped, Frequency-Mismatch	$\eta = 0$ $\omega_{char}^2 = \frac{K_s}{M} \neq \omega_{app}^2$	$y_{part}(t) = \frac{F_{app}\cos(\omega_{app}t)}{M(\omega_{char}+\omega_{app})(\omega_{char}-\omega_{app})}$
Undamped, Frequency-Matched	$\eta = 0$ $\omega_{char}^2 = \frac{K_s}{M} = \omega_{app}^2$	$y_{part}(t) = \frac{F_{app}t\sin(\omega_{app}t)}{2M\omega_{app}}$
Damped	$\eta > 0$	$y_{part}(t) = \frac{F_{app}\cos(\omega_{app}t+\phi_{lag})}{\sqrt{M^2(\omega_{char}^2-\omega_{app}^2)^2+\omega_{app}^2\eta^2 l_o^2}}$ $\phi_{lag} = \tan^{-1}\left(\frac{\omega_{app}\eta l_o}{M(\omega_{char}^2-\omega_{app}^2)}\right)$

The phenomenon of resonance can be observed as the driving frequency approaches the characteristic frequency.

MATHEMATICA® Example: Lecture-23

Visualizing Solutions for the damped-forced harmonic oscillator

1. Write a function that calculates the general solution to the non-homogeneous second-order solutions with forcing function set to $F_{app} = \cos(\omega_{app}t)$ for initial conditions that have a fixed displacement, but no initial momentum: $y(t=0) = 1$ and $y'(t=0) = 0$. Let the function have input parameters for the mass, viscosity, and characteristic frequency.

2. Note that for many numerical inputs for parameters, the solutions may have spurious small imaginary parts—this can be dealt with with the function Chop.

3. Visualize solutions for 20 or 30 cycles for various input parameters

Resonance can have catastrophic or amusing (or both) consequences:

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Figure 23-1: Picture and illustration of the bells at Kendall square. Many people shake the handles vigorously but with apparently no pleasant e®ect. The concept of resonance can be used to to operate the bells e±ciently Perturb the handle slightly and observe the frequencies of the the pendulums| select one and wiggle the handle at the pendulum's characteristic frequency. The amplitude of that pendulum will increase and eventually strike the neighboring tubular bells.
From Cambridge Arts Council Website:
http://www.cambridgeartscouncil.org/public_art_tour/map_11_kendall.html
Artist: Paul Matisse Title: The Kendall Band - Kepler, Pythagoras, Galileo Date: 1987
Materials: Aluminum, teak, steel
Handles located on the platforms allow passengers to play these mobile-like instruments, which are suspended in arches between the tracks, "Kepler" is an aluminum ring that will hum for ¯ve minutes after it is struck by the large teak hammer above it. "Pythagoras" consists of a 48-foot row of chimes made from heavy aluminum tubes interspersed with 14 teak hammers. "Galileo" is a large sheet of metal that rattles thunderously when one shakes the handle.

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Figure 23-2: The Tacoma bridge disaster is perhaps one of the most well-known failures that resulted directly from resonance phenomena. It is believed that the the wind blowing across the bridge caused the bridge to vibrate like a reed in a clarinet.
(Images from Promotional Video Clip from The Camera Shop 1007 Paci¯c Ave., Tacoma, Washington Full video Available http://www.camerashoptacoma.com/)