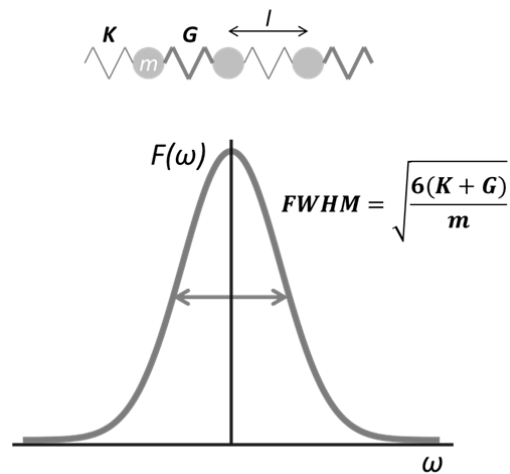


- (25 points) Tony Stark and Pepper Potts were trapped in adjacent cells separated by a 1 m thick wall of the hardest material known to man a 1:1 alloy of adamantium (Ad) and vibranium (Vb). Fortunately, the Golden Avenger has snuck in a mini-repulsor to communicate with his lady love to plan their daring escape using a previously agreed upon tapping code.

The repulsor generates a **Gaussian signal** that is immune to the dampening, but not immune to the effects of the **dispersion relation in diatomic lattice**. The frequency spectrum of the repulsor and a sketch of the Ad-Vb alloy structure are shown below. Vb and Ad share an atomic mass ( $m_{Ad}=m_{Vb} = 1 \times 10^{-25}$  kg) and bond length ( $l = 2 \text{ \AA}$ ), but the Ad-Ad bond is much stiffer ( $K = 1000 \text{ N/m}$ )<sup>a</sup> than the Ad-Vb bond ( $G = 100 \text{ N/m}^2$ ).



- (5 points) Sketch the time distribution  $f(t)$  of the pulse. Indicate what features are related to the FWHM of the frequency spectrum (you do not need to carry out precise calculations).

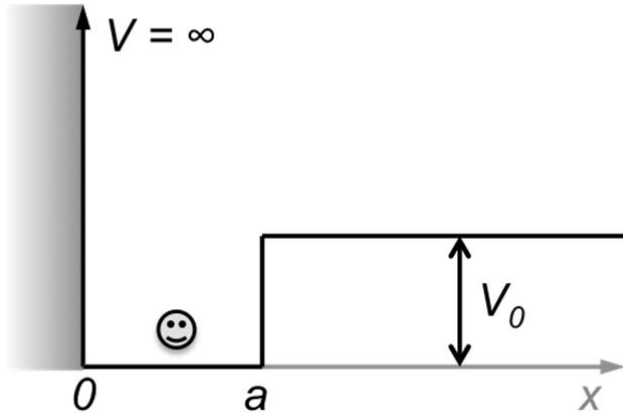
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<sup>a</sup>WOW!!!

b. (10 points) Sketch the frequency spectrum that will be heard by Ms. Potts on the other side of the wall. Explain.

c. (10 points) Time is short for our heroes! Calculate the fastest speed that signals can travel through the wall (remember  $K \gg G$ ). Will the whole signal travel at that speed? Why or why not?

2. (25 Points) Working in a hot Bay Area start-up, you have developed an ultrahigh-efficiency solar cell. Unfortunately, you have noticed an issue that prevents your device from reaching thermodynamic efficiency limits – the electrons are trapped at the surface ( $x = 0$ ) because of a surface-defect potential ( $0 < x < a$ ). You have measured the potential profile for the electrons and found  $V(x)$  below. As a solution, you deposited a very transparent material with a high perceived potential ( $V \rightarrow \infty$ ) on the surface ( $x > 0$ ) of your solar cell to reflect the electrons back into the cell.



$$V(x) = \begin{cases} V_0 = 35 \text{ meV}, & x > a \\ 0, & 0 < x \leq a \\ \infty, & x \leq 0 \end{cases}$$

- a) (8 points) The trapped electron starts out inside defect region ( $0 < x < a$ ) with energy  $E < V_0$ . Write the general form of the piecewise wave function and the relevant boundary and initial conditions.

b) (8 points) Based on your equations in part (a), find a relation between the electron wave vector inside the barrier and the defect region ( $0 < x < a$ ); find the relation between the corresponding wavefunction amplitudes.

c) (9 points) The chief technology officer of your startup suggested that you can collect the trapped electrons by heating the cell from your regular operation temperature  $T_1 = 300$  °K to  $T_2 = 500$  °K. Assuming that electron energy has a Boltzmann dependence on temperature  $E \approx k_B T$  (Boltzmann constant  $k_B = 1.4 \times 10^{-23}$  J/K =  $8.6 \times 10^{-5}$  eV/K) (very approximately) sketch the wavefunction of the electron at these two temperatures. (Assume  $V(x)$  doesn't change with  $T$ .) Explain how this method helps collect trapped electrons into the cell ( $x > a$ ).

3. (15 points) Before embarking on a dangerous quest against Oscillo, who aims to destroy the Earth using his armies of harmonites, together with Captain America, Dr. Bruce Banner managed to transmit the **code** to the Avenger's base in a **beam of electrons**. Unfortunately Dr. Banner was only able to tell the Captain that the code is:  $\langle x \rangle P_3 P_2 P_5 \langle E/E_0 \rangle$  before turning into a green rage monster. Now the Captain needs to decipher the code from the raw electron beam. Fortunately, he has a pocket diffractometer designed by Stark Industries. Using the device the Captain finds that the Dr. Banner's **electrons have a wavefunction of:**

$$\psi = e^{-m\omega x^2/2\hbar} \left( \frac{1}{2} H_1(x) + \frac{1}{2} H_3(x) + \frac{1}{\sqrt{2}} H_5(x) \right)$$

Where  $H_n(x)$  are Hermite polynomials.  $H_n(x) = (-1)^n e^{m\omega x^2/\hbar} \frac{d^n}{\left[ d \left( \sqrt{\frac{m\omega}{\hbar}} x \right) \right]^n} \left( e^{-m\omega x^2/\hbar} \right)$

Help our hero to escape into the base and save the unfortunate Dr. Banner by using your Quantum Mechanics superpowers:

- a. (5 points) Find the first number of the code  $\langle x \rangle$  – the expectation value of electron position.

b. (5 points) Find the probabilities  $P_2$ ,  $P_3$  and  $P_5$  to find the electrons in the **eigenstates 2, 3 and 5** of the **simple harmonic oscillator with a characteristic frequency  $\omega$ ; electron mass is  $m$** .

c. (5 points) Find the average energy of the electrons  $\langle E \rangle$  in the beam under the assumption that the electrons only experience a potential that can be approximated as  $V(x) = \frac{1}{2}m\omega^2 x^2$ . Find the last number in the code  $\langle E \rangle / E_0$ , where  $E_0$  is lowest energy eigenvalue for the Hamiltonian with the potential  $V(x)$ .

4. (35 points) During your internship at NASA your lab strikes a real scientific jackpot by discovering a part of a rare meteorite. Your preliminary examination confirms that the meteorite consists of a **periodic crystal**, but its atomistic nature is like nothing that you and your peers have ever seen. Now you need to figure out its properties – to accomplish this you pass electrons through the crystal and measure their momenta on the other side.

You find that with **equal probability** you get the following values of electron momentum:

$$p_1 = -\hbar \frac{3\pi}{a} \quad p_2 = -\hbar \frac{\pi}{a} \quad p_3 = \hbar \frac{3\pi}{a} \quad p_4 = \hbar \frac{5\pi}{a}, \quad a = 5 \text{ \AA}, \quad \hbar = 1.05 \times 10^{-34} \text{ J}\cdot\text{s}$$

- a. (5 points) What is the average value of momentum  $\langle \hat{p} \rangle$  for electrons in this structure? Is momentum conserved in this meteorite piece?

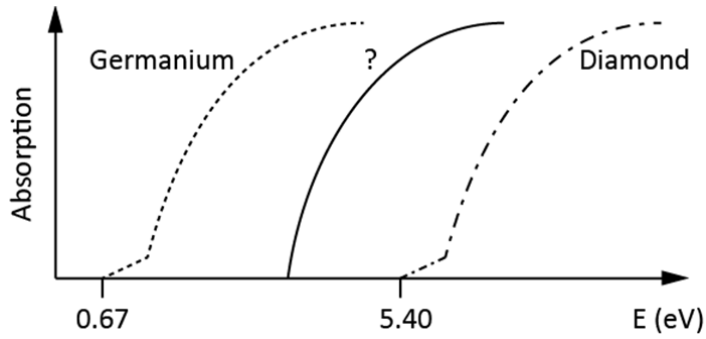
- b. (5 points) What is the wavefunction for electrons in this structure? Write it in terms of momentum operator eigenfunctions.

- c. (7 points) Express the wavefunction in the form:  $\psi(x) = e^{-i\frac{\langle \hat{p} \rangle}{\hbar}x} f(x)$ , where  $f(x)$  is a periodic function that you can express using sines and cosines.

- d. (3 points) Remembering the form of eigenfunctions in periodic potentials and using the wavefunction from (b) find the lattice constant of the extraterrestrial crystal.



In addition to your experiments with electrons, you were able to measure the **optical absorption** of this crystal and found that its spectrum lies between that of diamond and germanium (plot below).



e. (7 points) What can you say about the size of the atoms in this mystery material comparing it to diamond and germanium?

f. (8 points) Sketch the band diagram ( $E$  vs.  $k$ ) plot for this material at the band edge (edge of the Brillouin zone). How is it different from that of Germanium and diamond? (Qualitatively explain using  $E$  vs.  $k$  plots)?

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3.024 Electronic, Optical and Magnetic Properties of Materials  
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