3.032 Problem Set 4
Fall 2006
Due: Start of lecture, 10/20/06

1. Strain can be coupled not only to stress but also to thermal expansion or contraction (this is known as *thermoelastic coupling*). The general constitutive equation is $\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta T\delta_{ij}$, where α is the material's thermal expansion coefficient and ΔT is the change in temperature. For metals, α is on the order of $10\,\text{ppm/°C}$.
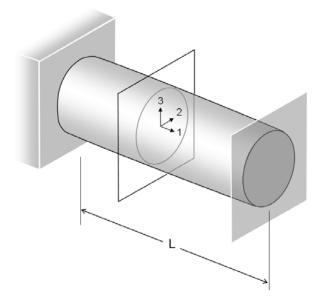

Figure 1: Bar of original length L in contact with two rigid walls.

(a) Consider the case of a long rod of length L that is in contact with two rigid walls and subject to a temperature increase ΔT (Figure 1). We are interested in the lateral strains—that is, the strains in the 2 and 3 directions. (Assumption: the walls do not limit strain in these directions.) Is the 2-3 plane in a state of *plane stress* or *plane strain*? Derive the value of $\varepsilon_{22} = \varepsilon_{33}$ as a function of α, ΔT, and the elastic constants.

(b) Find the change in volume ΔV of the bar, assuming small strains (*i.e.*, neglecting terms of order ε^2 and smaller). Compare this change in volume to the change in volume if the bar were not constrained.

(c) What is the change in volume ΔV if the Poisson's ratio ν of the constrained bar is zero? If ν is 0.5?

(d) What is the final diameter of a titanium rod of length $10\,\text{cm}$ and original diameter $4\,\text{mm}$ after a temperature increase of $250\,\text{°C}$? Use reasonable material properties for titanium. Would you expect the rod to buckle due to this temperature increase?

(e) Physically, what would you do in this experiment to ensure that the assumption in (a) is met?

2. Since there are only two independent elastic constants associated with isotropic materials, the shear modulus G is related to the Young's modulus E and the Poisson's ratio ν. Your goal is to derive this relationship.

(a) Consider a square region in a state of pure shear (Figure 2(a)). We expect the region to deform as shown in Figure 2(b). The shear modulus G is defined as the ratio of shear stress τ to the angular deformation γ (assuming that $\gamma \ll 1$). By what name is the variable γ known?
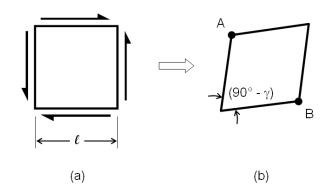

(a)   (b)

Figure 2: (a) Square region under a stress state of pure shear; (b) Deformed shape.

(b) Calculate the post-deformation distance between points A and B (or $\overline{AB}$) in terms of γ and ℓ. Feel free to use the small-angle approximations $\sin \gamma \approx \gamma$ and $\cos \gamma \approx 1$.

(c) Transform the stress state into the principal stress state by a method of your choosing. Plot the magnitudes and directions of the principal stresses on the original square region (Figure 2). Define a new set of axes, i' and j', corresponding to the directions of the principal stresses.

(d) Express the strain $\varepsilon_{i'j'}$ and also the post-deformation distance between points A and B ($\overline{AB}$) as a function of the principal stresses. By equating your two expressions for $\overline{AB}$, express G in terms of E and ν.

3. Listed below are force vs. strain data from tensile tests on one elemental polycrystalline metal, one polycrystalline metallic alloy, and one ceramic.

Force(N)	ε_A	ε_B	ε_C
0	0	0	0
500	0.00019	0.00017	0.00006
1000	0.00038	0.00035	0.00013
1500	0.00050	0.00054	0.00016
2000	fracture	0.00072	0.00025

(a) Graph these data and find the Young's elastic modulus E of each material, assuming an initial length of 16 cm and an initial cross-sectional area of 0.4 cm² for all samples.

(b) Which sample (A, B, or C) is likely to be the ceramic? Why?

(c) What is a possible composition of each metallic sample?

4. Materials (single crystals of metals and ceramics, composites, and even types of polymers and proteins) have symmetry that reduces the number of independent values in fourth rank tensors that modulate second rank tensors. You will prove this and consider the number of independent elastic constants in the context of elastic constants for cubic and isotropic materials.

(a) δ_{ij} is the Kronecker delta and $= 1$ (i = j) or $= 0$ (i ≠ j). We can write this as a matrix (not a tensor!), such that $\delta_{ij} =$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $a_{ij} = -\delta_{ij}$ and a_{ij} is the direction cosine matrix, as always. We can use this fact to write down some pretty obvious truths, e.g.,

$$\delta_{ij}\sigma_{jl} = \sigma_{il} \tag{1}$$

because, of course, $\sigma_{11} = 1 * \sigma_{11} + 0 * \sigma_{12} + 0 * \sigma_{13} = \sigma_{11}$. For this reason, the matrix δ_{ij} is also called the substitution matrix. Fascinating. Use this fact to prove something useful about the stiffness tensor C_{ijkl}:

$$C_{ijkl} = \delta_{im}\delta_{jn}\delta_{ko}\delta_{lp}C_{mnop} \tag{2}$$

(b) This means that a component of the stiffness and compliance tensors is unchanged in its value if the reference axes are rotated about a center of symmetry of the material. For example, a cubic materials has symmetry in the [100], [010], and [001] directions such that, in the contracted two-suffix notation of the fourth rank tensor C_{ijkl},

$$C_{11} = C_{22} = C_{33}; C_{12} = C_{23} = C_{31}; C_{44} = C_{55} = C_{66}. \tag{3}$$

To reduce from cubic symmetry (3 independent elastic constants) to isotropic symmetry (2 independent elastic constants), we can consider a rotation of the reference axes in a cubic crystal under uniaxial tensile strain ($\epsilon_{11} = \epsilon_{11}$; all other strains $= 0$). Express the normal stresses σ_{ij} in terms of ϵ_{ij} and C_{ijkl}, and then rewrite this expressing C in the contracted two-suffix notation.

(c) Consider a 45° CCW in-plane rotation from this old axis set (1, 2, 3) to a new axis set (1', 2', 3') about the 3-axis. Express the direction cosine matrix of this transformation a_{ij}.

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(d) Express the shear strain component ϵ'_{12} in terms of a_{ij} and the old coordinate system strains.

(e) Now express the shear stress σ'_{12} in terms of the old coordinate system stresses, and then in terms of the old axial strains that define those stresses.

(f) From (d) and (e) and the fact that you have proved that certain C_{ij} have the same value independent of rotation for materials of cubic symmetry, equate σ'_{12} with ϵ'_{12} in terms of C_{11} and C_{12}.

(g) Now, for an isotropic material, you know that shear stress is proportional to (engineering and tensorial) shear strain via the shear modulus. Equate σ'_{12} with ϵ'_{12} in terms of C_{44} on this basis.

(h) Finally, compare (f) and (g) to prove that three independent constants in a cubic material reduces to two independent constants for an isotropic material, and give the mathematical relationship among these three independent elastic constants C_{11}, C_{12}, and C_{44} for this isotropic approximation.

5. Most metals are elastically anisotropic, meaning that the measured Young's elastic modulus E_{ijk} depends on the direction with respect to the crystal structure along which load is applied. This can be expressed in terms of the compliance matrix S_{ij} as:

$$1/E_{ijk} = S_{11} - 2(S_{11} - S_{12} - \frac{1}{2}S_{44})(a_{i1}{}^2 a_{j2}{}^2 + a_{j2}{}^2 a_{k3}{}^2 + a_{i1}{}^2 a_{k3}{}^2) \qquad (4)$$

where this time a_{ij} is the direction cosine matrix between the unit cell crystallographic directions [100], [110], and [111]:

.	a_{i1}	a_{j2}	a_{k3}
[100]	1	0	0
[110]	$\sqrt{2}/2$	$\sqrt{2}/2$	0
[111]	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$

so one can compute E_{110} and compare it to E_{111}, for example. If all E_{ijk} are the same, that material is elastically isotropic. Referring to Table 2.4 (Meyers and Chawla) for S_{ij}, compute $E_{100}, E_{110}, and E_{111}$ as well as the Zener anisotropic elastic constants A for the following pairs of materials:

(a) Ag and Au, two FCC group 11 noble metals;

(b) W and Mo, two BCC group 6B heavy metals.

(c) Why do you think these differ *within* (a) and (b) [i.e., same crystal class and group]; and *among* (a) and (b)?