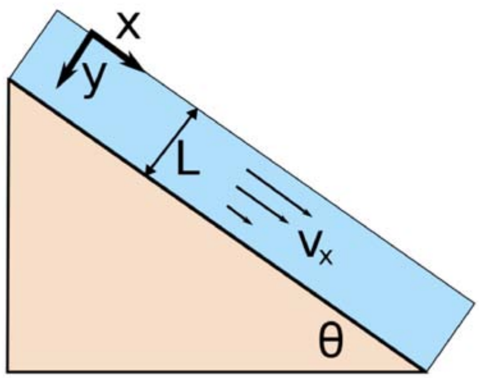
Pset 5 solutions
3.044 2013

Problem 1

First, let's identify the geometry:



and boundary conditions:

at y = 0, $\frac{dv_x}{dy} = 0$ (no shear)

at y = L, $v_x = 0$ (no slip).

For non-Newtonian fluids, $\tau_{yx} = -\mu \left(\frac{dv_x}{dy}\right)^m$.

When this is substituted into the momentum balalnce equation and we use $F_x = g\,\rho\,\sin(\theta)$, we see that

$\frac{\partial(\rho\, v_x)}{\partial t} = \frac{\partial}{\partial y}\left(\mu \left(\frac{dv_x}{dy}\right)^m\right) + g\,\rho\,\sin(\theta)$.

Note that when a non-Newtonian fluid is considered, the "m" appears inside the derivative with respect to y. Also, we are assuming steady-state, so $\frac{\partial(\rho\, v_x)}{\partial t} = 0$.

Rearranging:

$$\frac{\partial}{\partial y}\left(\nu \left(\frac{dv_x}{dy}\right)^m\right) = - g \sin(\theta)$$

and integrating:

$$\left(\frac{dv_x}{dy}\right)^m = -\ g/\nu\ \sin(\theta)\ y + A$$

Problem 2

Apply the no shear boundary condition:

$$\left(\frac{dv_x}{dy}\right)^m = -\ g/\nu\ \sin(\theta)\ y + A = 0 \text{ when } y = 0 \implies A = 0.$$

Rearrange, and integrate again:

$$\frac{dv_x}{dy} = (-g/\nu\ \sin(\theta)\ y)^{1/m}$$

$$v_x = \frac{1}{1/m+1}\left(\frac{-\nu}{g\sin(\theta)}\right)(-g/\nu\ \sin(\theta)\ y)^{1/m+1} + B$$

Apply the no slip bounary condition:

$$v_x = \frac{1}{1/m+1}\left(\frac{-\nu}{g\sin(\theta)}\right)(-g/\nu\ \sin(\theta)\ y)^{1/m+1} + B = 0 \text{ when } y = L$$

$$\therefore\ B = \frac{-1}{1/m+1}\left(\frac{-\nu}{g\sin(\theta)}\right)(-g/\nu\ \sin(\theta)\ L)^{1/m+1}$$

$$\boxed{v_x = \frac{1}{1/m+1}\left(\frac{g\sin(\theta)}{\nu}\right)^{1/m}\left(L^{1/m+1} - y^{1/m+1}\right)}$$

Note that when m = 1, this is the same as the Newtonian solution.

Problem 3

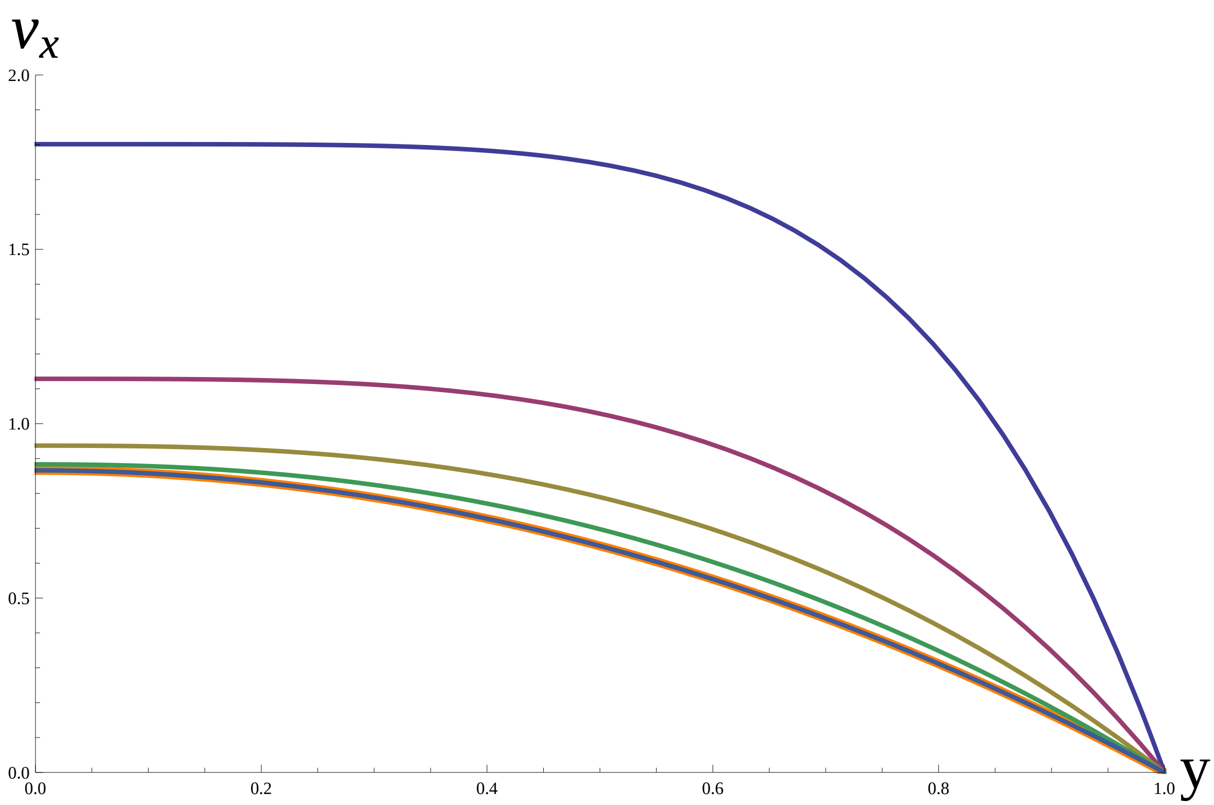
To plot the result, first define the function describing v_x:

```
vx[m_, y_, ν_] := (1 / ((1 / m) + 1)) (9.8 Sin[Pi / 4] / ν)^(1/m) (1 - y^((1/m)+1))
```

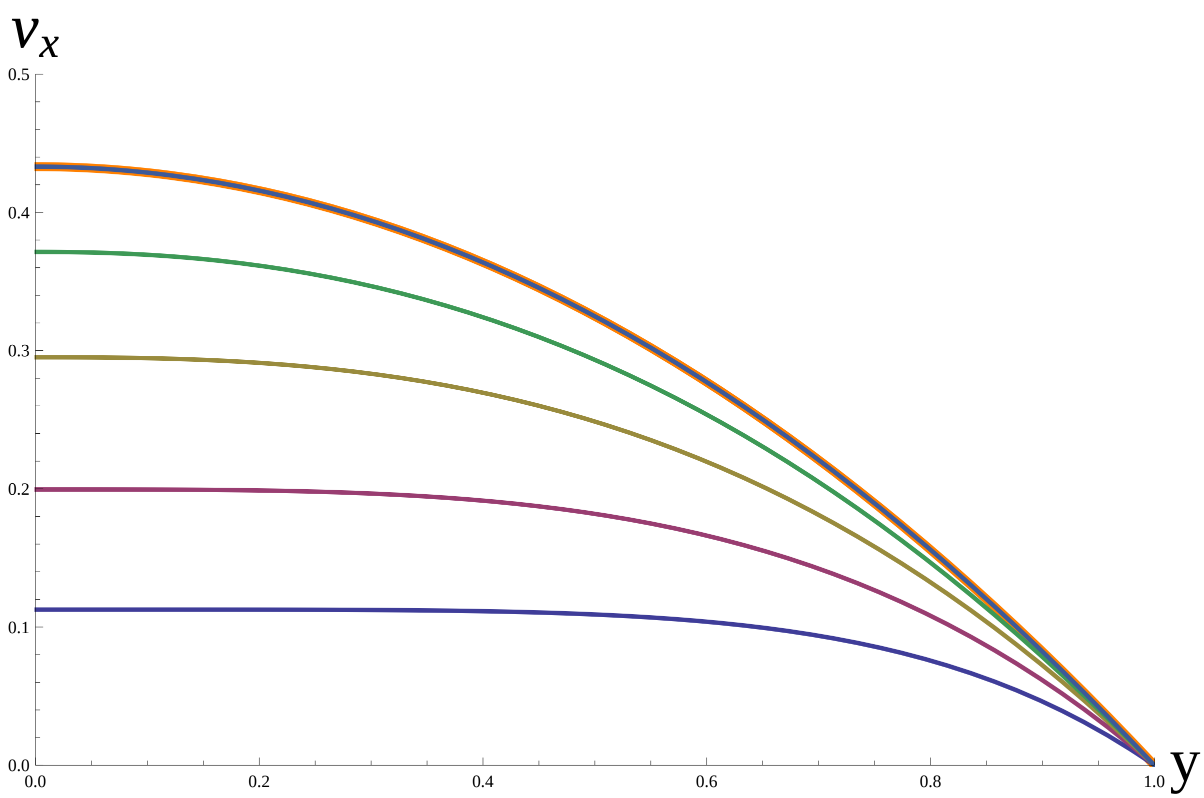
I chose g = 9.8, $\theta = \pi/4 = 45°$, and L = 1 (arbitrary)

Below is a plot with m = 0.2 (blue), 0.4 (red), 0.6 (yellow), 0.8 (green), and 1 (blue-gray). I also plotted the Newtonian solution we derived in class, in orange. You can see that the m = 1 solution and the Newtonian, parabolic solution completely overlap. The trend of whether the non-newtonian solutions are faster or slower than the Netonian solution depends on your choice of the magnitude of viscosity.

With ν = 4:

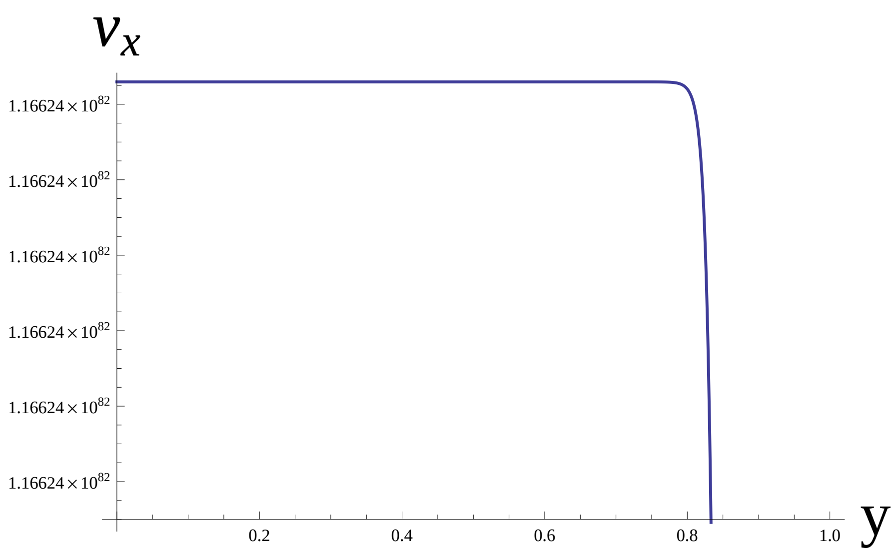


With ν = 8:



v_x
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.2
0.4
0.6
0.8
1.0
y

Either way, the non-Newtonian solution is “snubbier.” If you think about it, this make sense because the fluid has a lower viscosity where the shear, or $\frac{dv_x}{dy}$, is greater. Therefore, it is better to concentrate the gradients in a small range and have an area of high shear, but reduced viscosity (i.e., reduced resistance to shearing). The smaller m gets, the narrower this region becomes. In the limit of small m (see below: m = 0.01), the profile becomes square: the bulk of the fluid is actually acting like a solid. This almost-solid is gliding on a very thin layer of fluid that experiences essentially 100% of the shear required to satisfy the B.C.’s. This thin layer has dramatically lower viscosity because the shear is so high, making it very easy to flow.



Problem 4

I will assume that a typical water bottle is about 10 cm in diameter, and we are told that it must be 100 μm thick, so this has a volume of about (and ignoring the bottom of the bottle):
$V = 2\pi R t L$, with R = 0.05 m and L is the height of the bottle. We are assuming the bottle is thin relative to its diameter, and ignoring the bottom of the bottle for simplicity, but it would be correct to include it (it just turns out that the strains there are smaller, so that won’t be the point of failure). Also, we are assuming that the preform is the same length as the final bottle.

The volume of the preform is:
$V = \pi R_{out}^2 L - \pi R_{in}^2 L$, where R_{out} = the outer radius of the preform, which is comperable to that of a soda bottle cap, so about 1 cm in diameter, and R_{in} = the inner radius of the blank, which we must solve for.

The two volumes must be equal, so:
$\pi R_{out}{}^2 L - \pi R_{in}{}^2 L = 2\pi R t L$

Cancel the L's and π's, and rearrange to get:
$R_{in} = \sqrt{R_{out}{}^2 - 2 R t}$

Pugging in, we see that R_{in} = 0.949 cm, so the thickness of the blank must be about 0.05 cm = 500 μm thick.

Next, we are asked to find the fracture strains. In class, we found that $\epsilon_{fracture} = -m \ln\left(1 - \left(\frac{A_{o,i}}{A_{o,h}}\right)^{1/m}\right)$. We will take the ratio of areas to be 99%, since this is a typical value for a "routine" initial inhomogenaity, and the function is not especially sensitive to this ratio anyway. Plugging in the m for each material, we see that

$\epsilon_{fracture,\,5083}$= 272%

$\epsilon_{fracture,\,6061}$= 405%

$\epsilon_{fracture,\,7091}$= 139%

$\epsilon_{fracture,\,7475}$= 405%

In recitation, we learned that the maximum tensile strain experienced by a bottle being blow-molded is $\mathrm{Ln}\left(\frac{R_{bottle,\,inner}}{R_{preform,\,inner}}\right)$, which in our case is equal to ~166%, so we expect that Al 7091 will fail, Al 5083 has a decent chance at surviving, and both 6061 and 7475 are safe. However, for thinner bottles, or larger bottles, or more irregular preforms, even these nearly-Newtonian alloys would be at the cusp of failure.

Problem 5

Below is a plot of $(A_{0,i}/A_{0,h})$ versus groove depth as provided in the problem statement:

```
Plot[Exp[-290 x], {x, 0, 0.01}]
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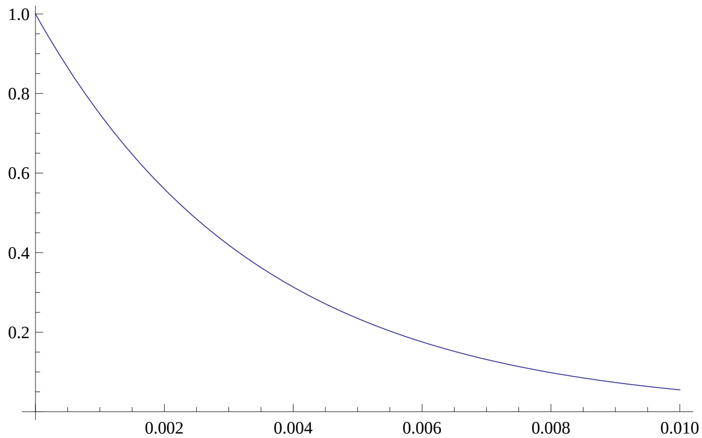

To find the groove depth value tolerable, plug in the above function for $(A_{0,i}/A_{0,h})$ in the fracture strain equaton, set that equal to the strain required to survive blow molding, and solve for "x," the groove depth:

```
In[54]:= m = 0.9;
         Solve[1.66 == -m Log[1 - Exp[-290 x]^(1/m)], x]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Out[55]= {{x → 0.000534133}}

(the error is harmless) for Al 6061 and Al 7475 which both have m = 0.9, the maximum groove depth is just 534 μm. Any deeper, and the inhomogeneity introduced by the gooves will cause rupture before the bottle reaches its full diameter. This is a very shallow groove!

```
In[56]:= m = 0.65;
         Solve[1.66 == -m Log[1 - Exp[-290 x]^(1/m)], x]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Out[57]= {{x → 0.000181492}}

Al 5083 has m = 0.65, so the maximum groove depth is just 181 μm.

```
In[58]:= m = 0.38;
         Solve[1.66 == -m Log[1 - Exp[-290 x]^(1/m)], x]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Out[59]= {{x → 0.0000167098}}

Al 7071 has m = 0.38, so it can only have 17 μm deep grooves. These would be imperceptable.

Now you can immagine why it was only in recent years that thin-walled metal

Now you can immagine why it was only in recent years that thin-walled metal bottles have become common, and why those “eco-shape” water bottles, which are super thin and have complicated shapes with lots of grooves to provide mechanical strength, were only introduced recently in plastic, which is very resistant to necking. Also, you see peraps why metal bottles do not typically have grooves on them.