3.185 Problem Set 7

Fluid Dynamics

Solutions

1. Power-law non-Newtonian fluid behavior

   (a) For pseudoplastic, the effective viscosity $-\tau/\dot{\gamma}$ decreases with increasing shear, so n must be less than one. (If one, it's constant; greater than one, it increases with increasing shear.)

   (b) Let's put $y = 0$ halfway between the plates, so the boundary conditions are $y = \pm\delta/2 \rightarrow u_x = 0$. Because we have symmetric boundary conditions and constant (and therefore symmetric) driving force and uniform properties, we can assume the velocity is symmetric about $y = 0$. For this reason, we can say that at $y = 0$, $\frac{du_x}{dy} = 0$, and see how this helps us. We can get $\frac{du_x}{dy}$ from above:

$$\frac{du_x}{dy} = \left(-\frac{\Delta P}{\mu_0 \cdot L} y + C_1'\right)^{\frac{1}{n}}$$

and set it to zero at $y = 0$:

$$0 = \left(-\frac{\Delta P}{\mu_0 \cdot L} \cdot 0 + C_1'\right)^{\frac{1}{n}}$$

$$0 = C_1'^{\frac{1}{n}}$$

So, $C_1' = 0$. What a relief! This simplifies the general solution to:

$$u_x = -\frac{\mu_0 \cdot L}{\Delta P} \frac{n}{n+1} \left(-\frac{\Delta P}{\mu_0 \cdot L} y\right)^{\frac{n+1}{n}} + C_2$$

and we can go a step further to:

$$u_x = \frac{n}{n+1} \left(-\frac{\Delta P}{\mu_0 \cdot L}\right)^{\frac{1}{n}} y^{\frac{n+1}{n}} + C_2$$

This is only strictly valid for non-negative $-\frac{\Delta P}{\mu_0 \cdot L} y$ and therefore for non-positive y. We can get the positive y velocities by symmetry.
Now use the boundary condition $y = -\frac{\delta}{2} \Rightarrow u_x = 0$ and solve for C_2:

$$0 = \frac{n}{n+1} \left(-\frac{\Delta P}{\mu_0 \cdot L}\right)^{\frac{1}{n}} \left(-\frac{\delta}{2}\right)^{\frac{n+1}{n}} + C_2$$

$$C_2 = -\frac{n}{n+1} \left(-\frac{\Delta P}{\mu_0 \cdot L}\right)^{\frac{1}{n}} \left(-\frac{\delta}{2}\right)^{\frac{n+1}{n}}$$

which can also be written

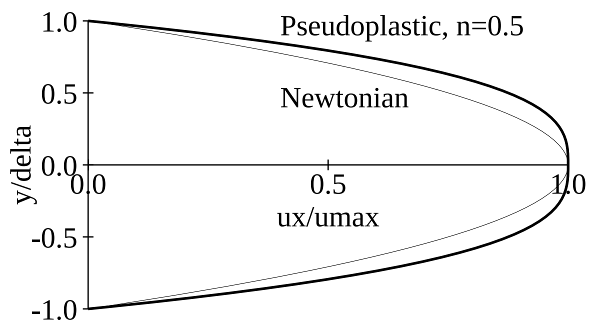
$$C_2 = \frac{n}{n+1} \frac{\delta}{2} \left(\frac{\Delta P}{\mu_0 \cdot L} \frac{\delta}{2}\right)^{\frac{1}{n}}$$

So the velocity profile for non-positive y is given by:

$$u_x = -\frac{n}{n+1}\left(-\frac{\Delta P}{\mu_0 \cdot L}\right)^{\frac{1}{n}}\left[\left(-\frac{\delta}{2}\right)^{\frac{n+1}{n}} - y^{\frac{n+1}{n}}\right]$$

Note that the maximum velocity at $y = 0$, which is given by C_2, is proportional to the thickness to the $1+\frac{1}{n}$ power. For a Newtonian fluid, this increases quadratically in thickness (*i.e.* proportional to thickness squared); for dilatant fluids ($n > 1$), between quadratic and linear; for pseudoplastic fluids ($n < 1$), faster than quadratic. Maximum velocity and flow rate are also nonlinear functions of pressure difference, pseudo-viscosity, and length!

(c) For $n = 0.5$, $u_x \sim y^3$, so this looks like:



(d) The velocity profile is more uniform around the maximum, so the average will be more than 2/3 of the maximum. In fact, for $n = 0.5$, it will be 3/4 of the maximum.

2. Extrusion of an aluminum tube

(a) In the annulus between the plug and the tube, we can assume:

- Incompressible laminar Newtonian flow with constant viscosity, because the problem says so for this part. This lets us use the simpler form of the equations.
- Steady-state, so time derivatives are zero.
- Axisymmetric flow, so the velocity derivatives with respect to θ are zero (though not necessarily pressure; if gravity were significant, then $\partial p/\partial\theta$ and $\partial p/\partial r$ would not be zero).
- Fully-developed flow, so velocity derivatives in the z-direction are zero, and flow is only in the z-direction (*i.e.* $u_r = u_\theta = 0$).

(b) When all of the terms are canceled, you should be left with:

$$\text{continuity}: \quad 0 = 0 \tag{1}$$

$$r\text{−momentum}: \quad 0 = -\frac{\partial p}{\partial r} \tag{2}$$

$$\theta\text{−momentum}: \quad 0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} \tag{3}$$

$$z\text{−momentum}: \quad 0 = -\frac{\partial p}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) \tag{4}$$

(c) For all three of these, we have the same stress profile, which is the same as the flux profile for cylindrical diffusion with generation:

$$\tau_{rz} = -\frac{1}{2}\frac{\partial p}{\partial z}r + \frac{A}{r}$$

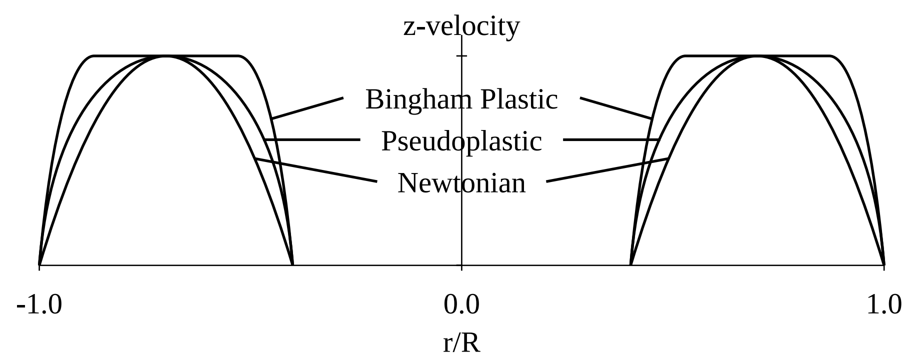
(where $\partial p/\partial z$ is negative). In the limit of a very thin annulus (so the plug nearly fills the tube), this is roughly linear like the stress distribution between parallel plates; we can use this simplified case to do the graphs.

i. For a Newtonian fluid, this gives a parabolic flow profile.

ii. For a pseudoplastic (shear-thinning) fluid, the magnitude of the shear stress is greatest near the walls, so the apparent viscosity is lowest there and highest in the center. This makes the velocity distribution flatter in the center, and more curved near the walls. (In problem set 7, you showed that for a power law pseudoplastic fluid with $n = 0.5$, the velocity distribution goes as $1 - y^3$ instead of the $1 - y^2$ parabolic Newtonian profile.)

iii. For a Bingham plastic, the lower shear stress near the center will lead to a region with $\partial u_z/\partial r = 0$, so the velocity profile is flat there. If the maximum shear stress is above the yield stress τ_0, then there will be a velocity gradient between this central solid region and the walls; otherwise the solid won't budge (unless there's some slip against the walls).

You were asked to graph these with the same maximum velocity, which should have looked like:



3. Torsional viscometer

(a) It's pretty clear that the cylindrical form is relevant. We're told to assume Newtonian flow with uniform density and viscosity, so we use the upper form of the Navier-Stokes equations given out in class on page 4 of the "Solving Fluid Dynamics Problems" handout, which is also the "D-F" form in Poirier and Geiger p. 59.

(b) Simplifying assumptions:

- Laminar flow: don't need to use Reynolds stresses.
- Steady-state: the viscometer runs until it reaches steady-state, then a measurement is taken, so it seems so. Time derivatives can be eliminated.
- Fully-developed: it is hard to define "fully-developed" for rotating flow. Instead, we'll say it's axially symmetric, so we can eliminate derivatives of velocity in the theta direction.
- Edge effects: Flow is in the θ-direction, so we have to choose "thickness" and "width" from the r- and z-directions. In the r-direction, the distance between the rod and cup is 3 cm, but the z-width of the flow is about 20 cm, so we'll choose r as the "thickness" direction where the velocity gradients are steep, and in the z-direction, the "edge" is the bottom of the viscometer, which we'll neglect for this analysis. So we can set the velocity derivatives in the z-direction to zero, and because centrifugal force is a function only of r, we can eliminate r- and z-components of velocity.
- Gravity is only in the z-direction, so we can ignore g_r and g_θ, and set g_z simply to the gravitational acceleration $-g$.

The equations with these eliminations simplify to the following, starting with continuity:

$$0 = 0 \text{ (everything cancels)}$$

Motion r-component:

$$\rho \frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r}$$

Motion θ-component:

$$0 = -\frac{\partial p}{\partial \theta} + \mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)$$

Motion z-component:

$$0 = -\frac{\partial p}{\partial z} - \rho g$$

That's much better than what we started with.

(c) Let's start by taking the derivative of the θ momentum equation with respect to θ to get the variation of pressure with respect to θ (as in class with the tube flow example we took the derivative of the z momentum equation with respect to z):

$$0 = -\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial}{\partial \theta}\left[\mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)\right]$$

We've assumed constant viscosity, and since r doesn't vary with θ and the mixed partials are equal, we can move the $\frac{\partial}{\partial \theta}$ derivative all the way into the viscous term:

$$0 = -\frac{\partial^2 p}{\partial \theta^2} + \mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_\theta}{\partial \theta}\right)\right)$$

Now we know that θ derivatives of velocity are zero, so that whole second term drops out and we're left with

$$0 = -\frac{\partial^2 p}{\partial \theta^2}$$

whose solution is

$$p = A(r,z)\theta + B(r,z)$$

Since the pressure is the same at $\theta = 0$ and $\theta = 2\pi$, the $A(r,z)$ must be zero, so $p = B(r,z)$, and $\frac{\partial p}{\partial \theta} = 0$. So we can get rid of that in the θ motion equation.
Now that equation is reduced to:

$$0 = \mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)$$

We can divide through by μ and integrate once to give

$$C(\theta, z) = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)$$

Then multiply by r, integrate again and divide by r:

$$C'(\theta, z) r + \frac{D(\theta, z)}{r} = v_\theta$$

where $C' = C/2$. Now because we've assumed fully-developed/symmetric flow without edge effects, v_θ is a function of r alone, so C' and D are constants rather than functions of r and z:

$$v_\theta = C' r + \frac{D}{r}$$

This is all we can do without boundary conditions.
[You could stop here, or continue if you liked.]
Next we can go to the z-direction:

$$0 = -\frac{\partial p}{\partial z} - \rho g$$

We move the pressure to the left side and integrate to give

$$p = -\rho g z + E(r)$$

where E is not a function of θ because we showed above that pressure doesn't depend on θ.
And to the r-direction:

$$\rho\frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r}$$

Substitute our v_θ from above:

$$\rho\frac{\rho}{r}\left(C'r + \frac{D}{r}\right)^2 = -\frac{\partial p}{\partial r}$$

Multiply it out:

$$\rho\left(C'r + \frac{2C'D}{r} + \frac{D}{r^3}\right) = -\frac{\partial p}{\partial r}$$

And integrate:

$$p = -\rho\left(\frac{C'r^2}{2} + 2C'D\ln r - \frac{D}{2r^2}\right) + F(z)$$

where F is again not a function of θ.
Now we equate the two pressure expressions:

$$\rho gz + E(r) = -\rho\left(\frac{C'r^2}{2} + 2C'D\ln r - \frac{D}{2r^2}\right) + F(z)$$

These can be equal if $F(z) = -\rho gz + G$ and $E(r) = -\rho\left(\frac{C'r^2}{2} + 2C'D\ln r - \frac{D}{2r^2}\right) + G$, ($G$ is a constant), so our pressure is given by

$$p = \rho\left(-gz - \frac{C'r^2}{2} - 2C'D\ln r + \frac{D}{2r^2}\right) + G$$

This gives us the shape of the liquid meniscus as a pressure isoquant (assuming we can neglect surface tension). Pretty cool!

(d) The θ-component of velocity is given by

$$v_\theta = C'r + \frac{D}{r}$$

At $r = R_1$, $v_\theta = \omega R_1$ (ω is the rotational speed in radians/second; some just used V as the boundary condition at R_1 and that's fine too), and at $r = R_2$, $v_\theta = 0$. Subtracting the two equations wouldn't help since there's no constant to cancel, so we'll multiply each by its radius then subtract:

$$R_1 \cdot \omega R_1 = R_1 \cdot \left(C'R_1 + \frac{D}{R_1}\right)$$

$$R_2 \cdot 0 = R_2 \cdot \left(C'R_2 + \frac{D}{R_2}\right)$$

$$\omega R_1^2 = C'(R_1^2 - R_2^2)$$

$$C' = \frac{\omega R_1^2}{R_1^2 - R_2^2}$$

Now we get D from the second boundary condition:

$$D = -C'R_2^2 = \frac{\omega R_1^2 R_2^2}{R_2^2 - R_1^2}$$

So the velocity profile is given by

$$v_\theta = \frac{\omega R_1^2}{R_2^2 - R_1^2}\left(\frac{R_2^2}{r} - r\right)$$

I suppose we could substitute this C' and D back into the pressure at the end of part 3c... Nah!

The torque is given by the integral of theta forces times radius. In this case, we integrate the shear stress-radius product $r(\tau_{r\theta} + \tau_{z\theta})$ over the inside surface of the outer cup. The problem states that we can neglect the shear stress $\tau_{z\theta}$ on the the bottom of the cup, and both τ_{rz} and r are constant on the cylindrical sides, so the torque is simply the product $r\tau_{rz} \cdot A$, so we use the torque from p. 61 of P&G:

$$\text{torque} = -r \cdot \mu r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) \cdot 2\pi r L$$

Substitute in the velocity from above:

$$\text{torque} = -2\pi r^3 L\mu \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\omega R_1^2}{R_2^2 - R_1^2}\left(\frac{R_2^2}{r} - r\right)\right]$$

Pull the constant out of the derivative, and note that the derivative of $\frac{r}{r}$ is zero:

$$\text{torque} = -\frac{2\pi r^3 L\mu\omega R_1^2}{R_2^2 - R_1^2}\frac{\partial}{\partial r}\left(\frac{R_2^2}{r^2}\right)$$

$$\text{torque} = \frac{2\pi r^3 L\mu\omega R_1^2}{R_2^2 - R_1^2}\frac{2R_2^2}{r^3}$$

The r^3 cancels, so the torque is independent of r:

$$\text{torque} = \frac{4\pi L\mu\omega R_1^2 R_2^2}{R_2^2 - R_1^2}$$

That the torque is independent of r is significant because it means the torque "in" equals the torque "out", so there is no angular momentum accumulation in the fluid, which confirms steady-state. This is analogous to the flux-area product's independence of r in diffusion.

(e) Solve the above torque expression for viscosity:

$$\mu = \text{torque} \cdot \frac{R_2^2 - R_1^2}{4\pi L\mu\omega R_1^2 R_2^2}$$

Substitute in given data:

$$\mu = 2.0 \times 10^{-3}\text{N}\cdot\text{m} \cdot \frac{(0.04\text{m})^2 - (0.01\text{m})^2}{4\pi \cdot 0.2\text{m} \cdot 120\frac{\text{rev}}{\text{min}} \cdot 2\pi\frac{\text{radians}}{\text{rev}} \cdot 60^{-1}\frac{\text{min}}{\text{sec}} \cdot (0.04\text{m})^2(0.01\text{m})^2}$$

$$\mu = 2.0 \times 10^{-3}\text{N}\cdot\text{m} \cdot \frac{1.5 \times 10^{-3}\text{m}^2}{5.05 \times 10^{-6}\frac{\text{m}^5}{\text{s}}}$$

$$\mu = 0.59\frac{\text{N}\cdot\text{s}}{\text{m}^2}$$

(f) According to the torque expression above, torque should be proportional to rotational speed. Since the torque doubles and the rotational speed increases by a factor of 2.5, this does not look like a Newtonian fluid. So the viscosity calculated above is only approximate.

The effective viscosity for this system is smaller at the higher strain rate, so the fluid is pseudo-plastic, which makes sense for a molten polymer.

4. Fluid flow in blood vessels

(a) As a rough ballpark estimate, say about 6 ml of blood is pumped with every heartbeat. At a pulse of 80 beats/minute, the flow rate is 8 $\frac{\text{cm}^3}{\text{s}}$.

(b) Average velocity is flow rate divided by cross-section area, which is $8\frac{\mathrm{cm}^3}{\mathrm{s}} \div [\pi(0.35\mathrm{cm})^2]$, or about 21 $\frac{\mathrm{cm}}{\mathrm{s}}$.

Reynolds number is $\frac{\bar{u}D}{\nu}$, $\bar{u} = 21\frac{cm}{s}$, $D = 0.7$cm, $\nu = 0.01\frac{cm^2}{s}$ for water, so it is about 1450. Flow is therefore laminar.

(c) For laminar flow, we can use the Hägen-Poiseuille equation for flow rate in a tube, derived in class:

$$Q = \frac{P_1 - P_2}{L}\frac{\pi}{8\mu}R^4.$$

We know the flow rate, viscosity and radius, so we can solve for the pressure gradient:

$$\frac{P_1 - P_2}{L} = \frac{8Q\mu}{\pi R^4},$$

and then using the force balance

$$\pi R^2(P_1 - P_2) = 2\pi RL\tau_{rz} \Rightarrow \tau_{rz} = \frac{P_1 - P_2}{L}\frac{R}{2},$$

we plug in the above equation for $(P_1 - P_2)/L$ to give:

$$\tau_{rz} = \frac{8Q\mu}{\pi R^4}\frac{R}{2} = \frac{4Q\mu}{\pi R^3}.$$

Plugging in our values $Q = 8\frac{\mathrm{cm}^3}{\mathrm{s}} = 8 \times 10^{-6}\frac{\mathrm{m}}{\mathrm{s}}$, $\mu = 10^{-3}\frac{\mathrm{N \cdot s}}{\mathrm{m}^2}$, $R = 0.035$m gives

$$\tau_{\mathrm{rz}} = \frac{4 \cdot 8 \times 10^{-6}\frac{\mathrm{m}^3}{\mathrm{s}} \cdot 10^{-3}\frac{\mathrm{N \cdot s}}{\mathrm{m}^2}}{\pi \cdot (0.0035\mathrm{m})^3} = 0.24\mathrm{Pa}.$$

You could also start with the friction factor (though that wasn't required, since we hadn't covered it in time), which is $\frac{16}{\mathrm{Re}} = 0.011$. Because we don't have the length, we can only approximate shear stress, which is given on p. 76 of P&G as

$$\tau_0 = fK = f\left(\frac{1}{2}\rho\bar{u}^2\right) = \frac{0.011}{2} \cdot 1000\frac{\mathrm{kg}}{\mathrm{m}^3} \cdot \left(0.21\frac{\mathrm{m}}{\mathrm{s}}\right)^2 = 0.24\mathrm{Pa}$$

If you used a larger ventricle volume or higher pulse rate it part 4a, the flow would be transitional or turbulent. In that case, you could use the friction factor analysis, though as mentioned in class, you were not responsible for knowing how to do that for this problem set, so you could leave the answer as "Can't do this yet".

(d) If flow is laminar, we can extend the equation for the pressure drop $P_1 - P_2$ derived above:

$$\frac{P_1 - P_2}{L} = \frac{8Q\mu}{\pi R^4} = \frac{8\bar{u}\mu}{R^2}.$$

With the parameters of this part of the problem $\bar{u} = 10^{-3}\frac{\mathrm{m}}{\mathrm{s}}$, $R = 2.5 \times 10^{-5}$m, $\mu = 0.001\frac{\mathrm{N \cdot s}}{\mathrm{m}^2}$, this gives us a pressure gradient of

$$\frac{P_1 - P_2}{L} = \frac{8 \cdot 10^{-3}\frac{\mathrm{m}}{\mathrm{s}} \cdot 0.001\frac{\mathrm{N \cdot s}}{\mathrm{m}^2}}{(2.5 \times 10^{-5}\mathrm{m})^2} = 12800\frac{\mathrm{Pa}}{\mathrm{m}}.$$

Multiplying by the length (1mm) gives 12.8 Pa for the pressure difference.

To use the friction factor, first calculate the Reynolds number which is $\frac{\bar{u}D}{\nu} = 0.05$, so flow is definitely laminar, and $f = 320$. This gives a shear stress of 0.16 Pa, and total drag force is this times the surface area of $1.57 \times 10^{-7}m^2$, so $F_d = 2.51 \times 10^{-18}N$, and $\Delta P = 12.8$ Pa.

(e) There are several important reasons why the above analysis will be inaccurate:

- The most important problem is that it assumes steady-state flow, which is obviously far from true.
- The tube flow analysis assumes rigid walls, whereas blood vessels are softenough to give slightly during each surge of pressure.
- The walls of blood vessels are not smooth, though this is much more of a factor for small blood vessels than large ones.
- For capillaries where blood cell size is a significant fraction of the diameter, the Newtonian flow assumption breaks down.
- For the aorta in particular, it is not a straight tube, but has a large bend, and dynamic pressure in that bend can be quite high. It is also short relative to the diameter, so fully-developed flow is questionable. And there are blood vessels branching off very early.

For those interested in further reading on fluid dynamics of blood flow, Jewell Wu (who took 3.185 in Fall 2000) found a very interesting website: `http://www.rwc.uc.edu/koehler/biophys/3e.html`