Chapter 4

Fluid Dynamics

4.1 October 24, 2003: Intro, Newtonian Fluids

TODO: look up Wiedmann-Franz, falling film in new textbook.
Mechanics:

- PS5: Get the spreadsheet from the Stellar site (URL on PS was for PMMA properties).

- L expression may be off: calc'd to 7.9×10^{-9}, not 2.45×10^{-8}. Maybe a missing π.

- Subra on bionano cell mechanics next Monday 4PM 10-250. Recruiting...

- Next Tues: MPC Materials Day, on Biomed Mat'ls Apps. Register: http://mpc-web.mit.edu/

Muddy from last time

- First "rule" in viewfactor algebra: $A_1 F_{12} = A_2 F_{21}$, doesn't it depend on T? No, because F_{ij} is based only on geometry.

- F_{12} for facing coaxial disks with radii r_1 and r_2 sample graph: F_{12} is decreasing with d/r_1.

Fluid Dynamics! Brief introduction to rich topic, of which people spend lifetimes studying one small part. You will likely be confused at the end of this lecture, come to "get it" over the next two or three.
Categories: laminar, turbulent; tubes and channels; jets, wakes. Compressible, incompressible.
Outcomes: flow rates (define), drag force (integral of normal stress), mixing. Later couple with diffusion and heat conduction for convective heat and mass transfer.
Start: the 3.185 way. Momentum field, "momentum diffusion" tensor as shear stress. Show this using units: momentum per unit area per unit time:

$$\frac{\text{kg}\frac{\text{m}}{\text{s}}}{\text{m}^2 \cdot \text{s}} = \frac{\text{kg}}{\text{m} \cdot \text{s}^2} = \frac{\text{N}}{\text{m}^2}$$

Two parallel plates, fluid between, zero and constant velocity. x-momentum diffusing in z-direction, call it τ_{zx}, one component of 2nd-rank tensor. Some conservation of math:

$$\text{accumulation} = \text{in} - \text{out} + \text{generation}$$

Talking about momentum per unit time, $\frac{\text{kg}}{\text{m} \cdot \text{s}^2}$, locally momentum per unit volume $\rho \vec{u}$. Here suppose u_x varies only in the z-direction, no τ_{xx} or τ_{yx}, no u_y or u_z. Three conservation equations for three components of momentum vector, here look at x-momentum:

$$V \cdot \frac{\partial(\rho u_x)}{\partial t} = A \cdot \tau_{zx}|_z - A \cdot \tau_{zx}|_{z+\Delta z} + V \cdot F_x$$

Do this balance on a thin layer between the plates:

$$W_x W_z \Delta z \frac{\partial (\rho u_x)}{\partial t} = W_x W_z \tau_{zx}\big|_z - W_x W_z \tau_{zx}\big|_{z+\Delta z} + W_x W_z \Delta z F_x$$

Cancel $W_x W_z$ and divide by Δz, let go to zero:

$$\frac{\partial (\rho u_x)}{\partial t} = -\frac{\partial \tau_{zx}}{\partial z} + F_x$$

What's generation? Body force per unit volume, like gravity. Units: N/m^3 (like τ has N/m^2), e.g. ρg.

What's the constitutive equation for τ_{zx}? Newtonian fluid, proportional to velocity gradient:

$$\tau_{zx} = -\mu \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right)$$

This defines viscosity μ, which is the momentum diffusivity. Units: $N \cdot s/m^2$ or $kg/m \cdot s$, Poiseuille. CGS units: $g/cm \cdot s$, Poise = 0.1 Poiseuille. Water: .01 Poise = .001 Poiseuille.

So, sub constitutive equation in the conservation equation, with $u_y = 0$:

$$\frac{\partial (\rho u_x)}{\partial t} = -\frac{\partial}{\partial z} \left(-\mu \frac{\partial u_x}{\partial z} \right) + F_x$$

With constant ρ and μ:

$$\rho \frac{\partial u_x}{\partial t} = \mu \frac{\partial^2 u_x}{\partial z^2} + F_x$$

It's a diffusion equation!

So at steady state, with a bottom plate at rest and a top plate in motion in the x-direction at velocity U, we have: a linear profile, $u_x = Az + B$.

Aaliyah tribute: innovative complex beats, rhythmic singing, great performing. Started at an early age, passing at age 21? 22? a couple of years ago in plane crash was a major music tragedy.

4.2 October 27: Simple Newtonian Flows

Mechanics:

- Aaliyah: okay, you might have heard of her, but didn't expect from a Prof. MTV...

- Wiedmann-Franz: new text doesn't offer any help (p. 204, no constant).

- Subra on bionano cell mechanics next Monday 4PM 10-250. Recruiting...

- Tomorrow: MPC Materials Day, on Biomed Mat'ls Apps. Too late to register though :-(

Muddy from last time:

- Is the velocity in the x-direction, or the z-direction? x-direction, but it is quite confusing, τ_{zx} is flux of x-momentum in z-direction. Momentum is a vector, so we have *three* conservation equations: conservation of x-momentum, y-momentum, z-momentum. This vector field thing is a bit tricky, especially the vector gradient.

- I left out: a Newtonian fluid exhibits linear stress-strain rate behavior, proportional. Lots of nonlinear fluids, non-Newtonian; we'll get to later.

Intro: may be confused after last time. This time do a couple more examples with confined flow, including one cylindrical, hopefully clear some things up.

From last time: parallel plates, governing equation

$$\rho \frac{\partial u_x}{\partial t} = \mu \frac{\partial^2 u_x}{\partial z^2} + F_x$$

$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial z^2} + \frac{F_x}{\rho}$$

We have the diffusion equation! ν is the momentum diffusivity, like the thermal diffusivity $k/\rho c_p$ before it. Note: units of momentum diffusivity $\nu = \mu/\rho$: $\frac{\text{kg/m·s}}{\text{kg/m}^3} = \text{m}^2/\text{s}$! Kinematic ($\nu$), dynamic ($\mu$) viscosities.

Note on graphics: velocities with arrows, flipping the graphs sideways to match orientation of the problem. Cases:

- Steady-state, no generation, bottom velocity zero, top U:

$$u_x = \frac{U}{L} z, \ \tau_{zx} = -\mu \frac{U}{L}$$

  Shear stress:

$$\tau_{zx} = -\mu \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) = -\mu \frac{U}{z}$$

  Thus the drag force is this times the area of the plate.

- Unsteady, no generation, different velocities from $t = 0$.

$$u_x = U \text{erfc} \left(\frac{L - z}{2\sqrt{\nu t}} \right), \tau_{zx} = -\frac{1}{2\sqrt{\nu t}} \frac{2}{\sqrt{\pi}} \exp \left(-\frac{(L - z)^2}{4\nu t} \right)$$

- New: steady-state, generation, like book's falling film in problem 4.15 of W^{3}R, with θ the inclination angle off-normal so $g_x = g \sin \theta$, z is the distance from the plane. The steady-state equation reduces to:

$$0 = \mu \frac{\partial^2 u_x}{\partial z^2} + F_x$$

$$u_x = -\frac{F_x z^2}{2\mu} + Az + B$$

45

BCs: zero velocity at bottom plate at $z = 0$, free surface with zero shear stress at $z = L$, $F_x = \rho g_x = \rho g \sin \theta$, result: B=0, get

$$u_x = \frac{g \sin \theta}{2\nu}(2Lz - z^2)$$

Shear stress:

$$\tau_{zx} = -\mu \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) = \rho g \sin \theta (L - z)$$

This is the weight of the fluid per unit area on top of this layer!

Shear stress as the mechanism of momentum flux, each layer pushes on the layer next to it. Think of it as momentum diffusion, not stress, and you'll get the sign right.

Flow rate: Q, volume per unit time through a surface. If width of the falling film is W, then flow rate is:

$$Q = \int_S \vec{u} \cdot \hat{n} dA = \int_{z=0}^{L} u_x W dz = \frac{W g \sin \theta}{\nu} \left[\frac{Lz^2}{2} - \frac{z^3}{6} \right] = \frac{W g \sin \theta}{\nu} \frac{L^3}{3}$$

Average velocity is Q/A, in this case Q/LW:

$$u_{av} = \frac{Q}{LW} = \frac{g \sin \theta L^2}{3\nu};$$

$$u_{max} = u_x|_{z=L} = \frac{g \sin \theta L^2}{2\nu}.$$

So average velocity is 2/3 of maximum for falling film, channel flow, etc.

4.3 October 29: 1-D Laminar Newtonian Wrapup, Summary

Mechanics:

- Next Weds 11/5: 3B Symposium!

Muddy from last time:

- Erf solution: why

$$u_x = U \operatorname{erfc}\left(\frac{L-z}{2\sqrt{\nu t}}\right)?$$

  As long as it's semi-infinite, it's an erf/erfc solution (uniform initial condition, constant velocity boundary condition), so this works for $t \leq L^2/16\nu$. For erfs, they can start at 0, or somewhere else (zinc diffusion couple), and go "forward" or "backward". Graph the normal way. Also:

$$u_x = U\left[1 + \operatorname{erf}\left(\frac{z-L}{2\sqrt{\nu t}}\right)\right]$$

- Weight of fluid: consider layer of fluid from z to L, it has force in x- and z-directions $F_g = V\rho g \sin\theta$ and $-\cos\theta$ respectively. In x-direction, shear goes the other way: $F_\tau = -\tau_{zx}A$. So force balance for steady-state (no acceleration):

$$V\rho g \sin\theta - A\tau_{zx} = 0 \Rightarrow \tau_{zx} = \frac{V}{A}\rho g \sin\theta = (L-z)\rho g \sin\theta.$$

  In z-direction, this is balanced by pressure:

$$P = P_{atm} + \rho g \cos\theta(L-z).$$

Nice segue into pressure-driven flows. Suppose fluid in a cylinder, a pipe for example of length L and radius R, P_1 on one end, P_2 on other. Net force: $(P_1-P_2)A_{xs}$, force per unit volume is $(P_1-P_2)V/A_{xs} = (P_1-P_2)/L$. Can shrink to shorter length, at a given point, force per unit volume is $\Delta P/\Delta z \to \partial P/\partial z$. This is the pressure generation term.

So, flow in tube: uniform generation throughout $(P_1 - P_2)/L$ (prove next week), diffusion out to $r = R$ where velocity is zero. Could do momentum balance, but is same as diffusion or heat conduction, laminar Newtonian result:

$$\rho\frac{\partial u_z}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\mu\frac{\partial u_z}{\partial r}\right) + \rho g_z - \frac{\partial P}{\partial z}.$$

Here looking at steady-state, horizontal pipe, uniform generation means:

$$u_z = -\frac{F_z r^2}{4\mu} + A\ln r + B = -\frac{P_1 - P_2}{4\mu L}r^2 + A\ln r + B.$$

Like reaction-diffusion in problem set 2 (PVC rod): non-infinite velocity at $r = 0$ means $A = 0$ (also symmetric), zero velocity at $r = R$ means:

$$u_z = \frac{P_1 - P_2}{4\mu L}(R^2 - r^2).$$

What's the flow rate?

$$Q = \int_0^R u_z 2\pi r dr = \int_0^R \frac{P_1 - P_2}{4\mu L}(R^2 - r^2)2\pi r dr$$

$$Q = \frac{\pi(P_1 - P_2)}{2\mu L}\left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_0^R = \frac{\pi(P_1 - P_2)R^4}{8\mu L}.$$

Hägen-Poisseuille equation, note 4th-power relation is extremely strong! 3/4" vs. 1/2" pipe...

Summary Summary of the three phenomena thus far:

	Diffusion	Heat conduction	Fluid flow
What's conserved?	Moles of each species	Joules of energy	momentum
Local density	C	$\rho c_p T$	$\rho \vec{u}$
Units of flux	$\frac{\text{mol}}{\text{m}^2 \cdot \text{s}}$	$\frac{\text{W}}{\text{m}^2}$	$\frac{\text{kg}\frac{\text{m}}{\text{s}}}{\text{m}^2 \cdot \text{s}} = \frac{\text{N}}{\text{m}^2}$
Conservation equation*	$\frac{\partial C}{\partial t} = -\nabla \cdot \vec{J} + G$	$\rho c_p \frac{\partial T}{\partial t} = -\nabla \cdot \vec{q} + \dot{q}$	$\frac{\partial(\rho \vec{u})}{\partial t} = -\nabla P - \nabla \cdot \tau + \vec{F}$
Constitutive equation	$\vec{J} = -D\nabla C$	$\vec{q} = -k\nabla T$	$\tau = -\eta \left[\nabla \vec{u} + (\nabla \vec{u})^T \right]$
Diffusivity	D	$\alpha = \frac{k}{\rho c_p}$	$\nu = \frac{\eta}{\rho}$

*Only considering diffusive fluxes. ^{T} in fluid constit. denotes the transpose of the matrix.

New stuff: vector field instead of scalar; very different units; pressure as well as flux/shear stress and force.

For those taking or having taken 3.11, shear stress τ relates to stress σ as follows:

$$\sigma = -\tau - PI$$

where p is $(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3$, so $\tau_{xx} + \tau_{yy} + \tau_{zz} = 0$, $\tau_{xy} = \tau_{yx} = -\sigma_{xy} = -\sigma_{yx}$. Note that $\tau_{yx} = \tau_{xy}$ almost always, otherwise infinite rotation...

Also: mechanics uses displacement for $\vec{u}$, acceleration is its *second* derivative with time. Simple shear:

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = \nabla \cdot \sigma + \vec{F} \Rightarrow \rho \frac{\partial^2 u_x}{\partial t^2} = \frac{\partial \sigma_{xx}}{\partial x^2} + \frac{\partial \sigma_{yx}}{\partial y^2} + \frac{\partial \sigma_{zx}}{\partial z^2} + F_x = G \frac{\partial^2 u_x}{\partial z^2} + F_x.$$

Analogue to momentum diffusivity: G/ρ, units m^2/s^2, $\sqrt{G/\rho}$: speed of sound! Remember with a little jig:

Fluids are diffusive,
With their velocity and viscosity.
But on replacement with displacement,
it will behave, like a wave!

4.4 October 31: Mechanics Analogy Revisited, Reynolds Number, Rheology

Mechanics:

- Next Weds 11/5: 3B Symposium!

- 3.21 notes on Stellar (with η changed to μ).

- In all of the viewfactor problems of PS6, ignore the graphs dealing with "nonconducting but reradiating" surfaces in the text, as we didn't cover those this year (or last year).

- In disc graph, not r_1/r_2 but D/r_1 and r_2/D.

- In problem 2, "collar" refers to the heat shield, a cylinder above the ZrO2 source.

- In problem 3, the fluid layer is thin enough that we can neglect the curvature of the ram and cylinder. The ram motion is also vastly powerful in terms of driving flow than the weight of the fluid, so we can neglect rho*g in the fluid. With these two simplifications, velocity profile between the cylinder and ram is linear, making the problem a lot simpler.

- In problem 4a, Arrhenius means proportional to $\exp(-\Delta G_a/RT)$. For part b, think about what happens to glasses as they cool...

- In problem 5, think about conduction through a multi-layer wall, in terms of the interface condition. Also, just as the multi-layer wall has two layers with different values of A and B in the temperature graph T=Ax+B, here we have two fluid layers with different values of A and B in the falling film solution given in class (that solution is the answer to part a). The Reynolds number will be discussed in tomorrow's lecture.

Muddy from last time:

- Crossbar on z.

- Why is $A = 0$, $B = \frac{P_1 - P_2}{4\mu L} R^2$ in tube? General solution:

$$u_z = -\frac{P_1 - P_2}{4\mu L} r^2 + A \ln r + B,$$

  boundary conditions: $r = 0 \rightarrow u_z$ not infinite, $r = R \rightarrow u_z = 0$. So, nonzero A in $A \ln r$ gives infinite u_z at $r = 0$, use B to exactly cancel first term at $r = R$.

  Note: at axis of symmetry, $\partial u_z/\partial r = 0$, like temperature and conc; this too would give $A = 0$.

- Mechanics analogy: very rushed, deserves better treatment, even though not a part of this class.

Reynolds number Like other dimensionless numbers, a ratio, this time of convective/diffusive momentum transfer, a.k.a. inertial/viscous forces. Formula:

$$\text{Re} = \frac{\rho U L}{\eta}$$

Describes dimensionless velocity for dimensionless drag force f; also onset of turbulence.

- Tubes: $< 2100 \rightarrow$ laminar (very constrained).

- Channel, Couette: $< 1000 \rightarrow$ laminar.

- Falling film (inclined plane): $< 20 \rightarrow$ laminar, due to free surface.

Note didn't give number for turbulent, that's because it depends on entrance conditions.

Rheology Typical Newtonian viscosities:

- Water: $10^{-3} \mathrm{N \cdot s/m^2}$, density $10^3 \mathrm{kg/m^3}$, kinematic viscosity (momentum diffusivity) $10^{-6} \mathrm{m^2/s}$.

- Molten iron: $5 \times 10^{-3} \mathrm{N \cdot s/m^2}$, density $7 \times 10^3 \mathrm{kg/m^3}$, kinematic viscosity (momentum diffusivity) just under $10^{-6} \mathrm{m^2/s}$, close to water. Water modeling...

- Air: $10^{-5} \mathrm{N \cdot s/m^2}$, density $1.9 \mathrm{kg/m^3}$, kinematic viscosity (momentum diffusivity) $5 \times 10^{-6} \mathrm{m^2/s}$, close to water!

But, very different effect on surroundings, drag force. So even though might flow similarly over a hill, geologists can tell the difference between water and wind erosion damage.

Liquids: generally inverse Arrhenius; gases (forgot).

Non-Newtonian: graphs of τ_{yx} vs. $\partial u_x / \partial y$. Categories:

- Dilatant (shear-thickening), example: fluid with high-aspect ratio solid bits; blood. More mixing, momentum mixing, acts like viscosity. Platelet diffusivity, concentration near walls...

  Model: power-law, $n > 1$.

- Pseudoplastic (shear-thinning), examples: heavily-loaded semi-solid, many polymers get oriented then shear more easily.

  Model: power law, $n < 1$.

  Next time: Wierd shear-thinning behavior at low strain rates due to fibrinogen content, extremely sensitive to fibrinogen and measures risk of cardiovascular disease better than smoking! (Gordon Lowe)

- Bingham plastic: finite yield stress, beyond that moves okay, but up to it nothing. Some heavily-loaded liquids, polymer composites, toothpaste; semi-solid metals bond together then break free.

  Model: yield stress τ_0, slope μ_P.

Power law relation:
$$\tau_{yx} = \mu_0 \left(\frac{\partial u_x}{\partial y} \right)^n$$

More than 1-D leads to wierd Tresca, von Mises criteria, etc.

4.5   November 3, 2003: Navier-Stokes Equations!

Mechanics:

- Weds 11/5: 3B Symposium!

- PS7 material last on Test 2... Due next Weds.

- Today: mid-term course evaluations!

Muddy from last time:

- In fluids, is τ a shear or normal stress? Equating it to σ is confusing... It's a shear stress. It might have "normal" components, but if you rotate it right, it's all shear.

- What's the difference between a liquid and a Newtonian fluid? Liquids are fluids, as are gases, and plasmas. Definition: finite (nonzero) shear strain rate for small stresses. Bingham plastic not technically a fluid, nor in a sense is a strict power-law pseudoplastic (though tends to break down near 0).

The Navier-Stokes equations! Pinnacle of complexity and abstraction in this course. From here, we explain, we see more examples, we fill in more details. Probably last time without a motivating process... So don't be surprised if you don't understand all of this just now, it's a bunch of math but should be clearer as we go on.

3.21 "Are you ready for momentum convection?"

Conservation of mass   2-D Navier-Stokes: two equations for three unknowns! Need one more equation, conservation of mass, only in-out by convective mass flux $\rho\vec{u}$, no mass diffusion or generation:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho\vec{u})$$

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla\rho + \rho\nabla \cdot \vec{u} = 0$$

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \vec{u} = 0$$

Incompressible definition: $D\rho/Dt = 0$. Example: oil and water, discuss $D\rho/Dt$ and $\partial\rho/\partial t$. Incompressible, not constant/uniform density. Result: $\nabla \cdot \vec{u} = 0$.

Eulerian derivation of Navier-Stokes   Like before; this time add convective momentum transfer. What's that? In convective mass transfer it was $\rho c_p \vec{u} T$. Now it's $\rho\vec{u}\vec{u}$. Wierd outer product second-rank tensor!

But what is momentum convection? Those $u_y \frac{\partial u_x}{\partial y}$ terms. We'll come back to those later. Recall heat transfer with convection:

$$\rho c_p \frac{\partial T}{\partial t} + \nabla \cdot (\rho c_p \vec{u} T) = -\nabla \cdot \vec{q} + \dot{q}$$

With fluids, it's the same:

$$\frac{\partial(\rho\vec{u})}{\partial t} + \nabla \cdot (\rho\vec{u}\vec{u}) = -\nabla p - \nabla \cdot \tau + \vec{F}$$

Left side expansion, simplification:

$$\rho\frac{\partial\vec{u}}{\partial t} + \vec{u}\frac{\partial\rho}{\partial t} + \rho\vec{u}\nabla \cdot \vec{u} + \rho\vec{u} \cdot \nabla\vec{u} + \vec{u}\vec{u} \cdot \nabla\rho = \vec{u}\left[\frac{\partial\rho}{\partial t} + \vec{u} \cdot \nabla\rho + \rho\nabla \cdot \vec{u}\right] + \rho\left[\frac{\partial\vec{u}}{\partial t} + \vec{u} \cdot \nabla\vec{u}\right]$$

Recall the continuity equation:

$$\frac{\partial\rho}{\partial t} + \vec{u} \cdot \nabla\rho + \rho\nabla \cdot \vec{u} = 0.$$

Entire first part cancels! Result:

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p - \nabla \cdot \tau + \vec{F}.$$

Now with Newtonian viscosity, incompressible:

$$\tau = -\mu \left(\nabla \vec{u} + \nabla \vec{u}^T - \frac{2}{3} \nabla \cdot \vec{u} I \right).$$

For x-component in 2-D:

$$\tau_x = -\mu \left[2 \frac{\partial u_x}{\partial x} \hat{i} + \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \hat{j} \right]$$

$$-\nabla \cdot \tau_x = \mu \left[2 \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_y}{\partial x \partial y} \right] = \mu \left[\nabla^2 u_x + \frac{\partial}{\partial x} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) \right]$$

So, that simplifies things quite a bit.

Incompressible Newtonian result:

$$\rho \frac{Du_x}{Dt} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u_x + F_x$$

Written out in 3-D:

$$\rho \left(\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} \right) = -\partial p / \partial x + \mu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right) + F_x$$

Identify (nonlinear) convective, viscous shear "friction" terms, sources.

The full thing in vector notation:

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \mu \nabla^2 \vec{u} + \vec{F}$$

Pretty cool. Again, sorta like the diffusion equation.

[For more on this, including substantial derivative, see 2000 David Dussault material, and "omitted" paragraphs from 2001.]

4.6 November 5, 2003: Using the Navier-Stokes Equations

Mechanics:

- Tonight: 3B Symposium! Dinner 5:30 in Chipman room.

- New PS7 on Stellar: no TODO, #4c graph on W^{3}R p. 188.

- Midterm Course Evals:

  - Lectures mostly positive, cards great (one: takes too long); negatives: 1/3 too fast, 1/2 math-intense; need concept summaries.
  - TA: split, most comfortable, 1/3 unhelpful, many: recs need more PS help. "Makes the class not hurt so badly."
  - PSes: most like, old PSes a prob, need more probs for test studying.
  - Test: like policy, but too long.
  - Text: few helpful different approach, most useless! But better...
  - Time: from 2-5 or 3+ to 12-18; "4 PS + 2-4 banging head against wall."

Muddy from last time:

- When we started momentum conservation, we had a P, but at the end, only ρ. Where did the P go? Still there. General equations:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} = 0$$

$$\rho \frac{D\vec{u}}{Dt} = -\nabla P - \nabla \cdot \tau + \vec{F}$$

  Incompressible, Newtonian, uniform μ:

$$\nabla \cdot \vec{u} = 0$$
$$\rho \frac{D\vec{u}}{Dt} = -\nabla P + \mu \nabla^2 \vec{u} + \vec{F}$$

- Fifth equation in compressible flows: $\rho(P)$, e.g. ideal gas $\rho = MP/RT$.

Convective momentum transfer   That nonlinear $D\vec{u}/Dt$ part that makes these equations such a pain! Example: $t = 0 \rightarrow u_x = 1$, $u_y = 2x$, sketch, show at $t = 0$ and $t = 1$. Convecting eddies in background velocity.

Using the Navier-Stokes Equations   Handout, start with full equations and cancel terms. Flow through tube revisited. Longitudinal pressure trick with z-derivative of z-momentum equation:

$$\frac{\partial^2 P}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{\mu}{r}\frac{\partial}{\partial r}\left(\frac{\partial u_z}{\partial r}\right)\right] = \frac{\mu}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2 u_z}{\partial r \partial z}\right) = 0! \ P = A(r,\theta)z + B(r,\theta)$$

$$0 = -\frac{\partial P}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(\frac{\partial u_z}{\partial r}\right) \Rightarrow u_z = \frac{\partial P/\partial z}{4\mu}r^2 + A\ln r + B = -\frac{P_1 - P_2}{4\mu L}z^2 + A\ln r + B.$$

Lateral pressure: if $\theta = 0$ points up:

$$\rho g_r = -\rho g \cos\theta = \partial P/\partial r,$$
$$\rho g_\theta = \rho g \sin\theta = (1/r)\partial P/\partial \theta.$$

Resulting pressure:

$$P = -\rho g r \cos\theta + f(z) = f(z) - \rho g h,$$

for h increasing in the upward direction from z-axis. Final result:

$$P = Az - \rho g h + C.$$

4.7 November 7, 2003: Drag Force

Mechanics:

- New PS7 on Stellar: no TODO, #4c graph on W^{3}R p. 188.

- Midterm Course Evals:

    - Lectures mostly positive, cards great (one: takes too long); negatives: 1/3 too fast, 1/2 math-intense; need concept summaries.

    - TA: split, most comfortable, 1/3 unhelpful, many: recs need more PS help. "Makes the class not hurt so badly."

    - PSes: most like, old PSes a prob, need more probs for test studying. Bad: last material in last lecture before due! This time: option Weds/Fri, okay?

    - Test: like policy, but too long, too long delay to retake.

    - Text: few helpful different approach, most useless! But better...

    - Time: from 2-5 or 3+ to 12-18; "4 PS + 2-4 banging head against wall."

- Test 2 in less than two weeks... Shorter wait for retake this time.

Muddy from last time:

- Redo pressure derivation.

$$\frac{\partial^2 P}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{\mu}{r}\frac{\partial}{\partial r}\left(\frac{\partial u_z}{\partial r}\right)\right] = \frac{\mu}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2 u_z}{\partial r \partial z}\right) = 0! \ P = A(r,\theta)z + B(r,\theta)$$

Lateral pressure: if $\theta = 0$ points up:

$$\rho g_r = -\rho g \cos\theta = \partial P/\partial r,$$

$$\rho g_\theta = \rho g \sin\theta = (1/r)\partial P/\partial\theta.$$

Resulting pressure:

$$P = -\rho g r \cos\theta + f(z) = f(z) - \rho g h,$$

for h increasing in the upward direction from z-axis. Final result:

$$P = Az - \rho g h + C.$$

- What's fully-developed, edge effects in cylindrical coordinates? Flow direction derivatives deal with fully-developed. If axisymmetric and flow is mostly θ, then can use this as "fully-developed".

    Edge effects: flow direction, large variation direction, third direction is "edge" direction. For tube flow, was θ, so axisymm is equiv to "no edge effects". For rod-cup, will be something else.

Drag force Integrated traction (force per unit area) in one direction. For tubes, it's in the z-direction, use the shear traction (stress) times area: $F_d = \tau \cdot A$. Here, looking at τ_{rz}:

$$u_z = \frac{P_1 - P_2}{4\mu L}(R^2 - r^2) \Rightarrow \tau_{rz} = -\mu\frac{\partial u_z}{\partial r} = \frac{P_1 - P_2}{2L}r.$$

$$F_d = 2\pi R L \cdot \tau_{rz}|_{r=R} = 2\pi R L \frac{P_1 - P_2}{2L}R = \pi R^2(P_1 - P_2).$$

Neat result: this is just the net pressure force. Okay, what if we know desired velocity, want to estimate required pressure? In terms of u_{av}:

$$u_{av} = \frac{u_{max}}{2} = \frac{(P_1 - P_2)R^2}{8\mu L},$$

$$F_d = \frac{(P_1 - P_2)R^2}{8\mu L} \cdot 8\pi\mu L = 8\pi\mu L u_{av}.$$

For any laminar flow, this is the drag force. Drag is shear traction times area:

$$F_d = \tau_{rz} \cdot 2\pi RL \Rightarrow \tau_{rz} = \frac{F_d}{2\pi RL} = \frac{8\pi\mu L u_{av}}{2\pi RL} = \frac{4\mu}{R}u_{av} = \frac{8\mu}{d}u_{av}.$$

Okay, using u_{av} to correlate with flow rate whether laminar or turbulent. So what about turbulent? Density plays a role because of convective terms. Dimensional analysis:

$$\tau = f(U, \mu, d, \rho)$$

Five parameters, three base units, so two dimensionless parameters. Four different nondimensionalizations!

Keep	π_τ	π_{other}
τ, μ	$\frac{\tau}{\rho U^2}$	$\pi_\mu = \frac{\mu}{\rho U d}$
τ, ρ	$\frac{\tau d}{\mu U}$	$\pi_\rho = \frac{\rho U d}{\mu}$
τ, U	$\frac{\tau d^2 \rho}{\mu^2}$	$\pi_U = \frac{\rho U d}{\mu}$
τ, d	$\frac{\tau}{\rho U^2}$	$\pi_d = \frac{\rho U d}{\mu}$

First and last are essentially the same, though last is more familiar (Reynolds number), so ignore the first. Third is just a mess, so throw it out. Second is a great fit to what's above. So use last, or second?

With turbulence, there are different curves with different surface roughnesses, roughly proportional to U^2. If use second, get one flat π_τ laminar, multiple lines for turbulent π_τ. If use first/last, one line for laminar, multiple flats for turbulance (p. 188). So this is generally more convenient.

Dimensionless π_τ is called (fanning) friction factor f (f_f). The denominator $\frac{1}{2}\rho U^2$ is the approximate kinetic energy density, we'll call it K, a.k.a. dynamic pressure. So:

$$\tau = fK, \ F_d = fKA, f = \frac{\tau}{\frac{1}{2}\rho U^2} = f(\text{Re}).$$

Laminar flow friction factor:

$$f = \frac{\tau}{\frac{1}{2}\rho U^2} = \frac{\frac{8\mu}{d}u_{av}}{\frac{1}{2}\rho U^2} = \frac{16\mu}{\rho U d} = \frac{16}{\text{Re}}.$$

To calculate drag force: Reynolds number (and surface roughness) $\rightarrow$ friction factor $f \rightarrow \tau = fK, F_d = fKA$.

4.8   November 12, 2003: Drag Force on a Sphere

Mechanics:

- Test 2 11/19 in 2-143, preview handout today.

- PS7 extended to Friday.

Muddy from last time:

- How is $f = 16\mu/\rho U d$?

$$\tau_{rz} = \frac{8\mu}{d}u_{av} \Rightarrow f = \frac{\tau_{rz}}{\frac{1}{2}\rho u_{av}^2} = \frac{8\mu u_{av}/d}{\frac{1}{2}\rho u_{av}^2} = \frac{16\mu}{\rho u_{av}d} = \frac{16}{\text{Re}}.$$

- What's the point of defining f? We just need τ, right? Well, f is easy for laminar flow, for turbulence it's more complicated. This gives us a parameter to relate to the Reynolds number for calculating drag force in more general situations. More examples on the way.

- Or was it $\vec{t}$, not f? In that case, traction $\vec{t} = \sigma \cdot \hat{n}$, force per unit area. In this case, with $n = \hat{r}$, $\vec{t} = (\tau_{rr} + p, \tau_{r\theta}, \tau_{rz})$. The relevant one for the z-direction is τ_{rz}, but traction is more general, see an example later today.

- Why was option #4 the "better" graph? Expanded version: with roughness at high Re, f is constant. So if Re=10^8, friction factor is *only* a function of e/d down to $e/d = 10^{-5}$!

- Please review this process $Q \rightarrow u_{av} \rightarrow \text{Re} \rightarrow f \rightarrow \tau \rightarrow F_d \rightarrow \Delta P$.

$$\frac{Q}{\pi R^2} = u_{av}, \ \text{Re} = \frac{\rho u_{av}d}{\mu}, \ f = f(\text{Re}, e/d), \ \tau_{rz} = fK, \ F_{d,z} = \tau_{rz}A = fKA(A = 2\pi RL), \ \Delta P = \frac{F_d}{\pi R^2}.$$

Reynolds Number revisited   Low velocity: shear stress; high velocity: braking kinetic energy. Ratio of forces:

$$\text{Re} = \frac{\text{convective momentum transfer}}{\text{shear momentum transfer}} = \frac{\text{inertial forces}}{\text{viscous forces}} \simeq \frac{\rho u_y \frac{\partial u_x}{\partial y}}{\mu \frac{\partial^2 u_x}{\partial y^2}} \simeq \frac{\rho UU/L}{\mu U/L^2} = \frac{\rho UL}{\mu}.$$

Flow past a sphere   Motivating process: Electron beam melting and refining of titanium alloys. Water-cooled copper hearth, titanium melted by electron beams, forms solid "skull" against the copper. Clean heat source, liquid titanium contained in solid titanium, results in very clean metal. Mystery of the universe: how does liquid Ti sit in contact with solid Cu? Main purpose: removal of hard TiN inclusions often several milimeters across which nucleate cracks and bring down airplanes! (1983 Sioux City, Iowa.)

Set up problem: sphere going one way u_{sphere}, fluid other way u_∞, local disturbance but relative velocity $U = u_\infty - u_{sphere}$, relative veloc of fluid in sphere frame. Drag force is in this direction.

For a sphere, drag force is slightly different: it has not only shear, but pressure component as well. Traction $\vec{t} = \sigma \cdot \hat{n}$. Stokes flow: ingore the convective terms, result (pp. 68-71):

$$F_d = 3\pi\mu du_{rel}.$$

At high velocity, similar friction factor concept to tube:

$$F_d = fKA = f \cdot \frac{1}{2}\rho U^2 \cdot \frac{1}{4}\pi d^2.$$

Again, $f(\text{Re})$, but not really π_τ because τ is all over the place, more of an average. This time, low Re (<0.1) means Stokes flow, can ignore all convective terms; analytical result in 3.21 notes, drag force:

$$F_d = 3\pi\mu Ud = f \cdot \frac{1}{2}\rho U^2 \cdot \frac{1}{4}\pi d^2 \Rightarrow f = \frac{24\mu}{\rho Ud} = \frac{24}{\text{Re}}.$$

56

If faster, though not turbulent, f becomes a constant. Graph on W^{3}R p. 153 of f (they call c_D) vs. Re. Constant: about 0.44, that's what I've known as the drag coefficient. Cars as low as 0.17, flat disk just about 1, making dynamic pressure a good estimate of pressure difference.

For rising/sinking particles, set drag force to buoyancy force, solve for velocity. If not Stokes flow, you're in trouble, way to do it but it's complicated.

Can *NOT* use this for bubbles. For those, $F_d = 2\pi\mu U d$ all the way out to Re=10^5!

So, what about precipitation? Buoyancy force and weight vs. drag force, all sum to zero:

$$F_w = \frac{1}{6}\pi d^3 \rho_{sphere}, F_b = \frac{1}{6}\pi d^3 \rho_{fluid}, F_d = \text{what we just calculated.}$$

After all, if you're not part of the solution, you're part of the precipitate. (Ha ha)

4.9 November 14, 2003: Boundary Layers Part I

Mechanics:

- Wulff Lecture Tues 4:15 6-120: *Information Transport and Computation in Nanometer-Scale Structures*, Don Eigler, IBM Fellow.

- Test 2 11/19 in 2-143. Solving Fluids Problems provided if needed.

- PS7 solution error: #1 replace L with δ, "length" with L. Correction on Stellar.

Muddy from last time:

- Which are the convective and viscous terms? In vector Navier-Stokes momentum:

$$\rho \frac{D\vec{u}}{Dt} = \rho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = -\nabla P + \mu \nabla^2 \vec{u} + \vec{F}.$$

  Convective terms are the $\vec{u} \cdot \nabla \vec{u}$ terms, like $u_x \partial u_y / \partial x$, the nonlinear terms which make the equations such a pain to solve and create turbulence... Viscous ones are $\mu \nabla^2 \vec{u}$.

- Why do thin ellipsoids have less drag than spheres, but flat plates have more? Depends on orientation. P&G has a few more examples on p. 87. Neat thing: the Stokes flow $f = 24/\mathrm{Re}$ holds for all of them!

- Are $\log(f)$ vs. $\log(\mathrm{Re})$ plots for only turbulent, or both laminar and turbulent? Both, that's the neat thing. For tube, sphere, and BL, it captures everything.

Sphere flow wrapup   Another neat way to think about Re: ratio of inertial to shear forces

$$\frac{\rho U^2 \cdot \pi d^2}{\mu d U} \propto \frac{\rho U d}{\mu} = \mathrm{Re}.$$

Can *NOT* use this flow-past-sphere stuff for bubbles. For those, $F_d = 2\pi \mu U d$ all the way out to Re=10^5! Boundary conditions...

"Boundary layers" in a solid   Thought experiment with moving solid: extruded polymer sheet (like PS4 extruded rod problem). Start at high temp, if well-cooled so large Biot then constant temperature on surface; no generation. Full equation:

$$\frac{DT}{Dt} = \alpha \nabla^2 T$$

Define boundary layer thickness δx where temperature deviates at least 1% from far-field. If $\delta \ll x$, then

$$\frac{\partial^2 T}{\partial y^2} >> \frac{\partial^2 T}{\partial x^2}$$

$$u_x \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2}$$

Transform: $\tau = x/u_x$, becomes diffusion equation, erf solution:

$$\frac{T - T_s}{T_i - T_s} = \mathrm{erf} \frac{y}{2\sqrt{\alpha x / u_x}}$$

If we define δ as where we get to 0.99, then $\mathrm{erf}^{-1}(.99) = 1.8$, and

$$y = \delta \text{ where } \frac{\delta}{2\sqrt{\alpha x / u_x}} = 1.8$$

$$\delta = 3.6 \sqrt{\frac{\alpha x}{u_x}}$$

Obviously breaks down at start $x = 0$, but otherwise sound.

Boundary layers in a fluid   Now we want to calculate drag force for flow parallel to the plate.

Similar constant IC to solid, infinite BC, call it U_∞. Difference: BC at $y = 0$: $u_x = u_y = 0$. Have to solve 2-D incompressible steady-state Navier-Stokes:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{\partial p}{\rho \partial x} + \nu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial x^2} \right)$$

$$u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} = -\frac{\partial p}{\rho \partial y} + \nu \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial x^2} \right)$$

Then a miracle occurs, the Blassius solution for $\delta \ll x$ is a graph of u_x/U_∞ vs. $\beta = y\sqrt{U_\infty/\nu x}$; hits 0.99 at ordinate of 5:

$$\delta = 5.0\sqrt{\frac{\nu x}{U_\infty}}.$$

Why 5.0, not 3.6? Because there must be vertical velocity due to mass conservation (show using differential mass equation and integral box), carries low-x-velocity fluid upward. Slope: 0.332, will use next time to discuss shear stress and drag force.

4.10 November 17, 2003: Boundary Layers Part II

Mechanics:

- Wulff Lecture Tues 4:15 6-120: *Information Transport and Computation in Nanometer-Scale Structures*, Don Eigler, IBM Fellow.

- PS7 solution error: #1 replace L with δ, "length" with L. Correction on Stellar.

- Zephyr hours tomorrow 9-12, 1-3; also Weds 4-7 PM.

- Test 2 11/19 in 2-143. Solving Fluids Problems provided if needed.

Muddy from last time:

- Nothing!

Recall our "miraculous" Blassius solution to the 2-D Navier-Stokes equations (draw the graph)... At $y = 0$, slope: 0.332, so viscous drag:

$$\tau_{yx} = -\mu \frac{\partial u_x}{\partial y} = -\mu \frac{\partial u_x}{\partial \beta} \frac{\partial \beta}{\partial y}$$

$$\tau_{yx} = -\mu \cdot 0.332 U_\infty \sqrt{\frac{U_\infty}{\nu x}}$$

Note: a function of x (larger near leading edge), diverges at $x = 0$! But $\delta \ll x$ does not hold there.

Now set to a friction factor:

$$\tau_{yx} = -0.332 \sqrt{\frac{\rho \mu U_\infty^3}{x}} = f_x \cdot \frac{1}{2} \rho U_\infty^2$$

This time τ is not constant, so we have different $f_x = \tau/K$ and $f_L = F_d/KA$. Let's evaluate both:

$$f_x = 0.664 \sqrt{\frac{\mu}{\rho U_\infty x}} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$$

Also, note dimensionless BL thickness:

$$\delta = 5.0 \sqrt{\frac{\nu x}{U_\infty}}$$

$$\frac{\delta}{x} = 5.0 \sqrt{\frac{\nu}{U_\infty x}} = \frac{5.0}{\sqrt{\mathrm{Re}_x}}$$

Lengthwise, global drag force, average friction factor. Neglect edge effects again...

$$F_d = \int \tau_{yx} dA = W \int_{x=0}^{L} tau_{yx} dx$$

$$F_d = W \int_{x=0}^{L} 0.332 \sqrt{\frac{\rho \mu U_\infty^3}{x}} dx$$

$$F_d = 0.332 W \sqrt{\rho \mu U_\infty^3} \cdot 2\sqrt{L}$$

$$F_d = 0.664 W \sqrt{\rho \mu U_\infty^3 L}$$

Now for the average friction factor/drag coefficient:

$$f_L = \frac{F_d}{KA} = \frac{0.664 W \sqrt{\rho \mu U_\infty^3 L}}{\frac{1}{2}\rho U_\infty^2 \cdot WL} = 1.328 \sqrt{\frac{\mu}{\rho U_\infty L}} = \frac{1.328}{\sqrt{\mathrm{Re}_L}}$$

This is what is meant by average and local friction factors on the Test 2 overview sheet. Don't need to know for test 2, since for a tube they're the same. For a sphere, only defined average/global, but for a BL, they're different.

Entrance Length   For channel flow between two parallel plates spaced apart a distance H, we can define the entrance length L_e as the point where the boundary layers from each side meet in the middle. The twin Blassius functions are close enough to the parabolic profile that we can say it's fully-developed at that point. So we can plug in the boundary layer equation if flow is laminar:

$$x = L_e \Rightarrow \frac{H}{2} = \delta = 5.0 \sqrt{\frac{\nu x}{U_\infty}},$$

$$L_e = \frac{H^2 U_\infty}{100 \nu}.$$

If $L_e \ll L$, then flow is fully-developed for most of the tube, so the fully-developed part will dominate the drag force and $F_d = \tau \cdot 2\pi R L$.

Movie Friday...

4.11 November 21, 2003: Turbulence

Mechanics:

- Turbulence movie: QC151.T8; guide QC145.2.F5 Barker Media.

Muddy from last time:

- What's the physical significance of "Blassius"? Some guy who came up with this function to solve a slightly-reduced 2-D Navier-Stokes. There are actually three, for u_x, u_y and p in the boundary layer, all $\delta \ll x$. For drag force, only u_x is needed.

- Is $dBL/d\beta$ only the slope at the initial part of the curve? If by "initial", you mean $y = 0$ (the bottom), then yes.

- What's the difference between f_x and f_L? Here, τ is not constant, graph τ and $\tau_{av} \to F_d$, show how average is twice local at the end. Hence f_L is twice f_x.

- How about "Re_x" and "Re_L"? No significance, just different lengthscales, one for local and one for global/average.

- Difference between u_{av} and U_∞? Should be no u_{av} for BL problems, sorry if I made a writo. D'oh! This is wrong, see next lecture's notes.

- What's up with δ/x? Just a ratio, dimensionless for convenience, allows to evaluate $\delta \ll x$.

- What is velocity profile for entrance length if not laminar? Get to that later (next Monday or so).

- What happens near leading edge? Something like the sphere: kinda complicated. Maybe solvable for Stokes flow...

Turbulence   Starting instability, energy cascade. Vortices grow in a velocity gradient because of momentum convection, damped due to viscosity; therefore, tendency increases with increasing Re.
Resulting behavior:

- Disorder.

- "Vorticity" in flow, 3-D.

- Lots of mixing, of mass and heat as well as momentum.

- Increased drag due to momentum mixing, as small vortices steal energy from the flow.

The movie!
Parviz Moin and John Kim, "Tackling Turbulence with Supercomputers," *Scientific American* January 1997 pp. 62-68.

> Turbulence may have gotten its bad reputation because dealing with it mathematically is one of the most notoriously thorny problems of classical physics. For a phenomenon that is literally ubiquitous, remarkably little of a quantitative nature is known about it. Richard Feynman, the great Nobel Prize-winning physicist, called turbulence "the most important problem of classical physics." Its difficulty was wittily expressed in 1932 by the British physicist Horace Lamb, who, in an address to the British Association for the Advancement of Science, reportedly said, "I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic."

That article goes on to talk about direct numerical simulation of all of the details of turbulent flows, of which I am not a great fan. Why? as pointed out in my *JOM* article "3-D or not 3-D", even if computers continue to double in computational power every eighteen months, the fifth-power scaling of complexity with lengthscale in three dimensions (cubic in space times quadratic in time) means that the resolution of these simulations will double only every seven-and-a-half years!

To take a simple illustrative example, direct numerical simulation of turbulence in continuous casting involves Reynolds numbers on the order of one million, and the smallest eddies are a fraction of a millimeter across and form and decay in a few milliseconds. One must account for interaction with the free surface, including mold powder melting and entrainment, as well as mold oscillation, and surface roughness of the solidifying metal, since with turbulent flow, the details of these boundary conditions can make a large difference in macroscopic behavior. The formulation alone is daunting, and computational work required to solve all of the equations on each of the tens of trillions of grid points over the millions of timesteps required to approach steady-state will be prohibitively costly for many years, perhaps until long after Moore's law has been laid to rest (indeed, the roughly four petabytes of memory required to just store a single timestep would cost about two billion dollars at the time this is being written). Furthermore, postprocessing that many degrees of freedom would not only be computationally difficult, but it is not clear that our minds would be able to comprehend the resulting complexity in any useful way, and further, the exercise would be largely pointless, as one really cares only about coarse-grained averages of flow behavior, and detailed behavior perhaps at certain interfaces.

Analysis: for $1 \frac{m}{s}$ flow through the nozzle 0.1 m in diameter, with kinematic viscosity of $.005/7000 \simeq 10^{-6}$ $\frac{m^2}{s}$, this gives us $\mathrm{Re}_d \simeq 10^5$. Using the larger lengthscale H of the caster, around 1 m, this gives $\mathrm{Re}_H \simeq 10^6$, this is given in the paragraph above. Using standard enengy cascade/Kolmogorov microscale analysis, the energy dissipation rate for the largest turbulent eddies in a tube is given by

$$\epsilon \sim \mu_t \left(\frac{U}{d} \right)^2,$$

where μ_t is the turbulent viscosity, U the eddy velocity estimated by the average velocity, and d the eddy size estimated by the tube diameter. Combining this with the well-known Kolmogorov result for the smallest eddy lengthscale ℓ:

$$\ell \sim \sqrt[4]{\frac{\mu^3}{\rho^2 \epsilon}}$$

(ρ is density) gives the smallest eddy lengthscale as

$$\ell \sim d \frac{1}{\sqrt{\mathrm{Re}_d}} \sqrt[4]{\frac{\mu}{\mu_t}}.$$

Even using the conservative estimate of $\mu_t = 30\mu$ gives $\ell \sim 0.1$ mm, in 1 meter cubed this gives a trillion grid points, but you want a few grid points across each smallest eddy which means about a few$^3 \simeq 100$ times more grid points, hence "tens of trillions".

Put slightly differently, the total rate of energy dissipated in a jet at steady-state is the rate of kinetic energy input, which is the product of volumetric kinetic energy given by dynamic pressure and the flow rate:

$$\epsilon V = \frac{1}{2} \rho U^2 \cdot \frac{\pi}{4} d^2 U = \frac{\pi}{8} \rho U^3 d^2$$

Taking the volume as 1 m^3, average velocity of $1 \frac{m}{s}$, nozzle diameter of 0.1 m and density of $7000 \frac{kg}{m^3}$ gives $\epsilon \simeq 30 \frac{W}{m^3}$. Putting this into the Kolmogorov lengthscale expression again gives $\ell \sim 0.1$ mm.

4.12 November 24, 2003: Turbulence, cont'd

Mechanics:

- Turbulence movie: QC151.T8; guide QC145.2.F5 Barker Media.

Muddy from last time:

- Difference between u_{av} and U_∞? I messed up last time, for the entrance length situaiton, we we take average velocity as U_∞, the initial "free stream" velocity. Sorry to confuse you last time.

- On that note, just as we have the dimensionless boundary layer thickness, we also have the dimensionless entrance length:

$$L_e = \frac{H^2 u_{av}}{100\nu} \Rightarrow \frac{L_e}{H} = \frac{\text{Re}_H}{100}.$$

  So a large Reynolds number means a long entrance length (1000 means ten times the channel width), and vice versa.

Energy cascade and the Kolmogorov microscale. Largest eddy Re$=UL/\nu$, smallest eddy Reynolds number $u\ell/\nu \sim 1$. Energy dissipation, W/m^3; in smallest eddies:

$$\epsilon = \eta \left(\frac{du}{dx}\right)^2 \sim \eta \frac{u^2}{\ell^2}$$

Assuming most energy dissipation happens there, we can solve these two equations, get smallest eddy size and velocity from viscosity, density and dissipation:

$$u \sim \ell \sqrt{\frac{\epsilon}{\eta}} \Rightarrow \frac{\rho\ell^2}{\eta} \sqrt{\frac{\epsilon}{\eta}} \sim 1$$

$$\ell \sim \left(\frac{\eta^3}{\rho^2\epsilon}\right)^{\frac{1}{4}}$$

This defines the turbulent microscale. For thermal or diffusive mixing, turbulence can mix things down to this scale, then molecular diffusion or heat conduction has to do the rest. Time to diffusive mixing in turbulence is approximately this ℓ^2/D.

So suppose we turn off the power, then what happens? Smallest eddies go away fast, then larger ones, until the whole flow stops. Timescale of smallest is ℓ^2/ν, largest is L^2/ν_t, turbulent effective viscosity. Get into modeling and structure later if time is available.

Turbulent boundary layer   Laminar is good until $\text{Re}_x = 10^5$, associated boundary layer thickness and local friction factor:

$$\frac{\delta}{x} = \frac{5.0}{\sqrt{\text{Re}_x}}, \; f_x = \frac{0.664}{\sqrt{\text{Re}_x}}, \; f_L = \frac{1.328}{\sqrt{\text{Re}_L}},$$

In range 10^5 to 10^7, transition, oscillatory; beyond 10^7 fully turbulent. Always retains a laminar sublayer against the wall, though it oscillates as vortices spiral down into it. New behavior:

$$\frac{\delta}{x} = \frac{0.37}{\text{Re}_x^{0.2}}$$

So $\delta \sim x^{0.8}$. Grows much faster. Why? Mixing of momentum, higher effective velocity. But still a laminar sublayer near the side.

f_x is oscillating all over the place, but what about the new f_L? Disagreement, even for smooth plate:

$$\text{P\&G p. 38}: f_L = \frac{0.455}{(\log \text{Re}_L)^{2.58}}; \; \text{BSL p. 203}: f_L = \frac{0.146}{\text{Re}_L^{0.2}}$$

Either way get some kind of curve in f-Re space which jumps in turbulence. W^{3}R doesn't give an f_L, just an f_x on p. 179, which is:

$$f_x = \frac{0.0576}{(\text{Re}_x)^{0.2}}.$$

Time-smoothing Time smoothing, or experiment-smoothing and Reynolds stresses: velocity varies wildly, decompose into $u_x = \bar{u}_x + u'_x$, where $\bar{u}$ is time-smoothed:

$$\bar{u}_x = \frac{\int_{t_a}^{t_b} u_x dt}{t_b - t_a}.$$

For time-dependent, experiment-average it.

Contest Prizes Wednesday for those who catch the two errors in the Navier-Stokes equations t-shirt!

Not covered this year The following topics were not covered in lecture, but are here for your edification if you're interested.

Reynolds stresses Then take time-smoothed transport equations:

$$\frac{\partial(\bar{u}_x + u'_x)}{\partial t} = \frac{\partial \bar{u}_x}{\partial t} + \frac{\partial \bar{u}'_x}{\partial t} = \frac{\partial \bar{u}_x}{\partial t} + 0.$$

Same with spatial derivatives, pressure terms. But one thing which doesn't time-smooth out:

$$u'_x \frac{\partial u'_x}{\partial x} \neq 0.$$

This forms the Reynolds stresses, which we shift to the right side of the equation:

$$\tau_{xy} = -\mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) - \rho \bar{u'_x u'_y}$$

Show how it's zero in the center of channel flow, large near the sides, zero *at* the sides. The resulting mass equation is the same; x-momentum equation:

$$\rho \left(\frac{\partial \bar{u}}{\partial t} + \bar{\vec{u}} \cdot \nabla \bar{u}_x \right) = -\frac{\partial \bar{P}}{\partial x} + \mu \nabla^2 \bar{u}_x - \rho \left(\frac{\partial}{\partial x}(\rho u'_x u'_x) + \frac{\partial}{\partial y}(\rho u'_x u'_y) + \frac{\partial}{\partial z}(\rho u'_x u'_z) \right) + F_x.$$

Turbulent transport and modeling Recall on test: pseudoplastic, Bingham. Define effective viscosity: shear stress/strain rate.

On a micro scale, lots of vortices/eddies. On a macro scale, mixing leads to higher effective D_t, k_t, and η_t at length scales down to that smallest eddy size, less so close to walls. All three turbulent diffusivities have the same magnitude.

New dimensionless number: Prandtl number is ratio of ν to diffusivity, e.g. ν/D and ν/α. $\mathrm{Pr}_t \simeq 1$ for heat and mass transfer.

Next time: thermal and solutal boundary layers, heat and mass transfer coefficients, turbulent boundary layer, then natural convection. Last, Bernoulli equation, continuous reactors.

Modeling: $K - \ell$ and $K - \epsilon$ modeling (C_μ, C_1, C_2, σ_K and σ_ϵ are empirical constants):

$$K = \frac{1}{2}\rho(u'^2_x + u'^2_y + u'^2_z), \nu_t = C_\mu \frac{K^2}{\epsilon}.$$

$$\frac{DK}{Dt} = \nabla \cdot \left(\frac{\nu_t}{\sigma_K} \nabla K \right) + \nu_t \nabla \vec{u} \cdot \left(\nabla \vec{u} + (\nabla \vec{u})^T \right) - \epsilon.$$

$$\frac{D\epsilon}{Dt} = \nabla \cdot \left(\frac{\nu_t}{\sigma_\epsilon} \nabla \epsilon \right) + C_1 \frac{\nu_t \epsilon}{K} \nabla \vec{u} \cdot \left(\nabla \vec{u} + (\nabla \vec{u})^T \right) - C_2 \frac{\epsilon^2}{K}.$$