3.22 Mechanical Properties of Materials
Spring 2008

Problem Set #3

Due: Friday, March 21 by 5:00 PM

1. Because metallic glasses have an amorphous atomic structure, many researchers believe they cannot "strain harden" like their crystalline counterparts (a topic we will cover in depth later). At present, you can understand this to mean that the stress required to continue deformation of the material to strains greater than the yield strain $\epsilon(\sigma_y)$ increases monotonically, often modeled as a power-law of the form $\sigma = K\epsilon_p^n$ where ϵ_p is the plastic strain, n is the material strain hardening exponent, and K is a material constant related to σ_y. In other words, after yielding the material becomes "harder (i.e. more difficult) to strain." Regardless of this assumption, you decide to have a UROP in your group conduct some experiments to consider whether strain hardening does occur in a particular metallic glass you have synthesized. You have the UROP compress a cylindrical sample of bulk metallic glass that is 6 mm long and 3 mm in diameter to failure. After the tests, the UROP excitedly presents the data to you (which can be found in 3-1.xls), claiming to have proven with these data that strain hardening in metallic glasses is possible! As she hyperventilates talking about the *Nature* article she wants to write, you realize that she is wrong (most likely because she did not have the opportunity to yet take 3.22).

   (a) Explain why she is wrong. Do this by calculating and discussing the $\sigma - \epsilon$ response(s) and the strain hardening exponent n for the material. Compare your results with what is expected of a material that does strain harden.

   Solution: We can plot these data as engineering and true stress strain. Engineering stress $\sigma_{\rm e}$ and strain $\epsilon_{\rm e}$ can be easily calculated from the given data using the initial cross-sectional area $A_\circ$ and length $l_\circ$ of the specimen such that

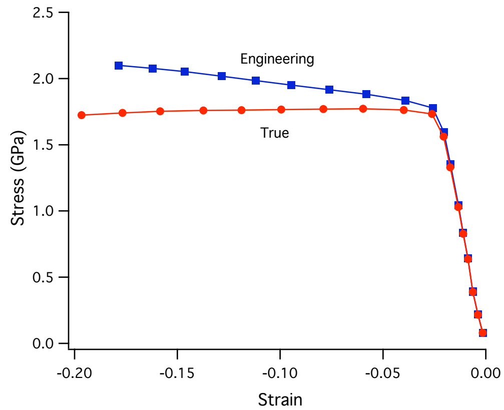
$$\sigma_{\rm e} = \frac{P}{A_\circ}$$
$$\epsilon_{\rm e} = \frac{-d}{l_\circ}$$

   where d is the displacement. The true stress $\sigma_{\rm t}$ and strain $\epsilon_{\rm t}$ can then be calculated by

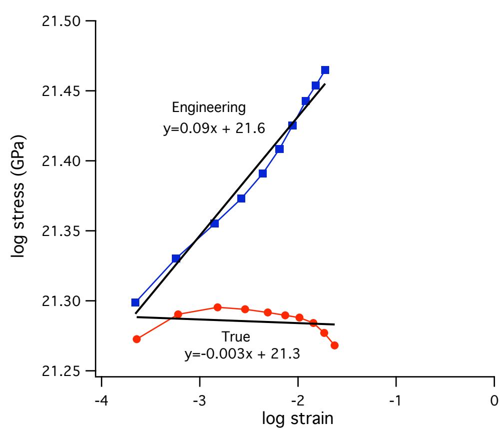
$$\sigma_{\rm t} = \sigma_{\rm e}(1+\epsilon_{\rm e})$$
$$\epsilon_{\rm t} = \ln(1+\epsilon_{\rm e}).$$

   The figure below is the plot of the true and engineering stress-strain curves based on the given data. Notice that the plastic regions in each curve are different. In the engineering stress-strain curve, the plastic region appears to have a positive slope suggesting some level of strain hardening while in the true stress-strain curve that material appears to act as perfectly plastic (e.g. no strain hardening).

   A measure of the strain hardening in a material is the strain hardening exponent n



given the equation $\sigma = K\epsilon_p^n$. This exponent can be determined by plotting the log of stress against the log of strain in the plastic region. The slope of a line fitted to these data yields n. For our data, the graphs and fits are shown below. We find that n in the engineering stress-strain data is ~ 0.09 and nominally zero in the true stress-strain data. Therefore we conclude that the UROP must have analyzed the engineering stress-strain data and thought she observed strain hardening in the material. To avoid this error, the true stress-strain data should always be utilized when actual properties of the material are calculated, as these measures of stress and strain more accurately reflect the deformation state of the material (not just what is easiest to measure)



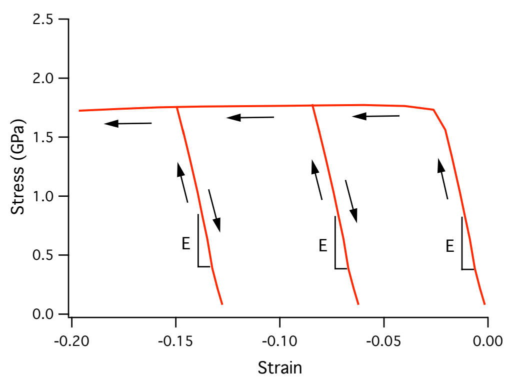
(b) Determine the elastic and plastic strain at failure.

Solution: The results of this problem depends on whether you used the true or engineering stress-strain data. The method is the same for each, but since we just lectured about the importance of using true stress-strain, we will stick with that.

The total strain ϵ_{tot} is the sum of the elastic strain ϵ_{el} and ϵ_{p}. From our data, we *find* $\epsilon_{\text{tot}} = -0.19$. *The elastic strain is just the strain accumulated in the elastic region up to the yielding point, which in our data occurs at* $\epsilon_{\text{el}} \simeq -0.3$. *Therefore* $\epsilon_{\text{p}} = (-0.19) - (-0.03) = -0.16$.

(c) Not one to give up easily, you think that you may be able to get the metallic glass to strain harden by applying a cyclic load (with the minimum load being 0 N) through the plastic region. Show what the resulting stress-strain curve would look like if you applied three cycles before failure.

Solution: When a material is loaded into the plastic region (but before failure) and then the load is released, the material will recover all the elastic strain (in most cases). The plastic strain however is permanent deformation and therefore the strain at zero applied stress will be equal to the plastic strain. If the material is then loaded again, it will initially strain elastically (following σ/E*) until the yield stress is reached and then the strain will be plastic. See the figure below.*



2. You are given a material and told that it is elastically isotropic and exhibits a yield strength σ_y of 950 MPa. The material is being used in an application where it experiences a stress state of

$$\sigma_{ij} = \begin{vmatrix} 0 & 0 & 300 \\ 0 & -400 & 0 \\ 300 & 0 & -800 \end{vmatrix} \text{MPa}$$

(a) According to the von Mises and Tresca criteria, respectively, does the material yield?

Solution: Before calculating the von Mises and Tresca stresses, I like to determine the principal stresses. You can do this using Mohr's circle (as was discussed in recitation) or by equations given in Courtney. From either method, we find

$$\sigma_P = \begin{vmatrix} 100 & 0 & 0 \\ 0 & -400 & 0 \\ 0 & 0 & -900 \end{vmatrix} \text{MPa}$$

The von Mises criterion is given by

$$\sigma_\circ = \frac{1}{2}\left[(\sigma_{\rm I} - \sigma_{\rm II})^2 + (\sigma_{\rm II} - \sigma_{\rm III})^2 + (\sigma_{\rm III} - \sigma_{\rm I})^2\right]$$

which yields $\sigma_\circ = 866$ *MPa. Since* $\sigma_\circ < \sigma_{\rm y} = 950$ *MPa, the material will not yield according to the Von Misses criterion.*

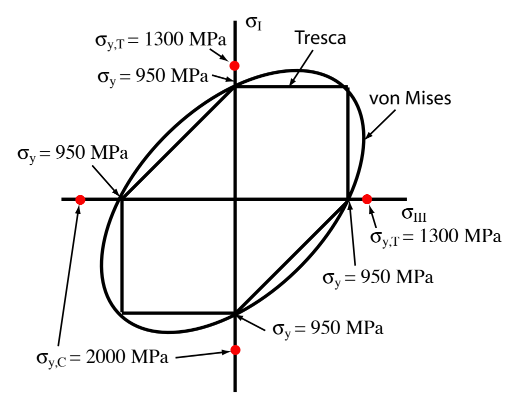
The Tresca criterion is

$$\sigma_\circ = \sigma_{\rm I} - \sigma_{\rm III}$$

and yields $\sigma_\circ = 1000$ *MPa, which is larger than* $\sigma_{\rm y}$*. Therefore according to the Tresca criterion, the material would fail. Note that we get two different answers.*

(b) Plot the yield surfaces (on the same graph) of the material based on the von Mises and Tresca criteria.

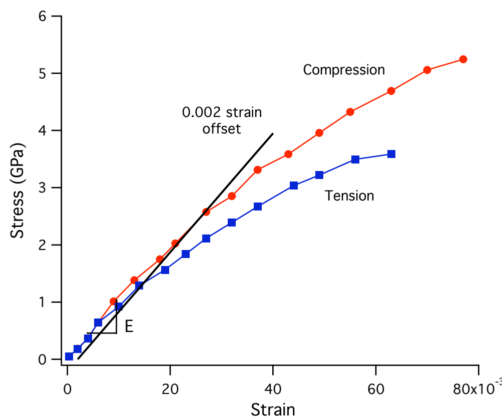
Soution: See the figure below. For simplicity, only a slice of the yield surface is shown. Accurate representation of each surface in three dimensional can be found under "Yield Surfaces" on Wikipedia and MechE-geared textbooks on continuum plasticity. In order to draw the entire surface, we must assume that the material is isotropic (an inherent assumption in von Mises and Tresca), and thus the yield stress of the material is the same in tension and compression.



(c) You decide to examine the material more closely by performing your own uniaxial tension and compression tests. The stress-strain data for these tests are found in 3-2.xls. Plot these data on the same graph as you made in (b). Discuss how the von Mises and Tresca criteria predict yielding in the material. Is there a better criterion to use for this material? Explain.

Solution: The yield stress in tension and compression can be determined from a line drawn with a strain offset of 0.002 and slope equal to the Young's Modulus (see figure below). Where this line intersects the strain-stress curve is the the yield stress.

Doing this we find the yield stress in tension $\sigma_{\mathrm{y,T}} = 1.3$ GPa *and in compression* $\sigma_{\mathrm{y,C}} = 2$ GPa*. These points are plotted on the graph in the solutions to part (b).*
We see that both the von Mises and Tresca criteria do not accurately predict the asym-



metry in the yielding stress observed in the tests. A yield criterion that accounts for this asymmetry should be used. Examples include the Mohr-Coulumb and Drucker-Prager criteria that are discussed in more detail in the following problems.

The data used in this problem were taken from a paper by A.C. Lund and C.A. Schuh (Acta Mat. **51***, 2399 (2003)) where they simulated yielding in metallic glass to examine the yield surface. In their analysis, they choose to use the Mohr-Coulumb criterion because they were only interested in the non-shear terms that contributed to yielding.*

3. Noting that the von Mises and Tresca criteria deviate for specific stress states in the yield surface considered above, determine the following:

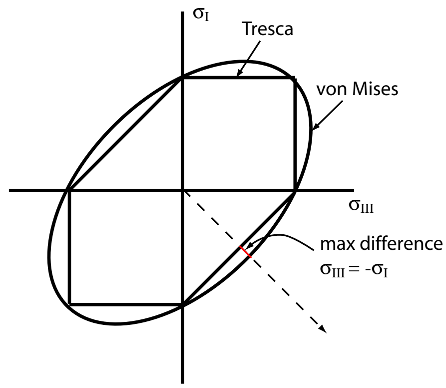
   (a) The stress tensor σ_{ij} corresponding to the stress state(s) at which there is maximal difference between these two yield criteria, expressed as a matrix and as a representative volume element (RVE) of the material. This will correspond to the biggest gap(s) between these two predictions in the graphical yield surface.

   Solution: The maximum difference between the von Mises and Tresca surfaces occur when $\sigma_1 = -\sigma_2$ *or* $\sigma_2 = -\sigma_1$ *(see the figure below). From Mohr's circle, this state is equivalent to one of pure shear or*

$$\sigma_{\mathrm{ij}} = \begin{vmatrix} 0 & 0 & \tau \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} \sigma_{\mathrm{I}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sigma_{\mathrm{III}} \end{vmatrix} \text{ MPa}$$

   (b) The magnitude of this difference, expressed first in terms of an algebraic expression including the σ_y of the material and second in terms of a percentage basis [%].

   Solution: For this state, Tresca criterion gives $\sigma_{\mathrm{eff}} = \sigma_{\mathrm{y}}/2$ *and von Mises gives* $\sigma_{\mathrm{eff}} =$



$\sigma_{\mathrm{y}}/\sqrt{3}$. *This is a 13% difference.*

(c) *Which* of these two criteria more accurately predicts the stress states corresponding to material yielding in pure metals and metallic alloys, and *why* does this tend to be the more accurate predictor? To answer this, you will need to rely on and show/cite experimental data for selected materials.

Solution: Considering several published yield surfaces, it is clear that von Mises better predicts yield in ductile metals that fail via shear. This is because all three stress axes play a role in deforming the lattice and contributing to the magnitude of shear stresses on slip planes.

4. The yield strength σ_y of many materials is poorly described by von Mises and Tresca; these include porous materials, glassy polymers, granular media, and biological materials. For such materials, σ_y depends in part on the state of hydrostatic stress σ_{ii} and thus the first invariant of σ_{ij}, I_1. Here, you will explore some of these pressure-sensitive yield criteria.

   (a) Mohr and Coulomb discussed continuum mechanics and particle interactions, respectively. Their ideas enabled modification of the yield criteria to reflect the fact that σ_y may differ in uniaxial tension vs. compression. This criterion is commonly used for bulk metallic glass and soils, which fail at lower applied stresses in tension than in compression. It linearly relates the effect of superposed normal and shear stresses on the stress state required to yield the material, and Mohr's circle helps us easily understand this. Assume an imposed principal stress state in a material plane; Mohr-Coulomb expresses the yield criteria in terms of the stress state at maximal shear stress resulting from σ_1 and σ_3, and can be expressed as the material yielding when:

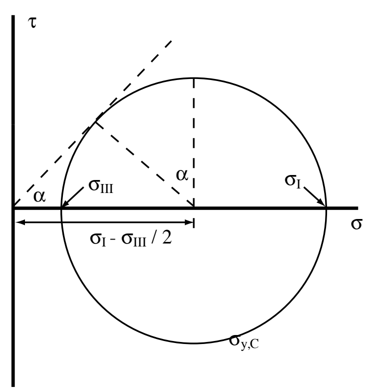
$$\tau = <\sigma> \sin\ \alpha \tag{1}$$

   where $<\sigma>$ is the average normal stress corresponding to τ, and $(\tau, <\sigma>)$ is the coordinate on Mohr's circle defined at the point of max shear stress for defined $\sigma_1 > 0$ and

$\sigma_3 > 0$, and α is the friction angle. Physically, α is the angle that would naturally arise if you poured the granular material onto a surface due to friction between the particles, and is usually between 15^o and 50^o. In terms of Mohr's circle, it is the angle between the vertical line connecting $(\tau, < \sigma >)$ to the $\tau = 0$ horizontal axis, and the tangent of the circle that intersects the point $(\tau = 0, \sigma = 0)$. Draw this Mohr's circle, and restate the Mohr-Coulomb yield criterion only in terms of these normal principal stresses and the material friction angle.

Solution: See the figure below for the Mohr's circle. Note that, more generally, the tangent to this circle need not intersect ($\sigma = 0$, $\tau = 0$*); it needs only cross the line defined as* $\sigma = 0$*, and then the y-intercept defines a finite k, the value of* τ *defined as the cohesion strength of the material. The Mohr-Coulomb yield criterion is thus restated as*

$$(\sigma_{\rm I} - < \sigma >)^2 + (\sigma_{\rm III} - < \sigma >)^2 = 2 < \sigma >^2 \sin^2 \alpha$$



(b) Drucker and Prager developed the creatively named Drucker-Prager yield criterion originally for soil mechanics and it is often applied to composite materials such as cement or concrete. It accounts for normal and shear stresses, and can be stated as:

$$f = \sqrt{J_2} - \alpha I_1 - k \tag{2}$$

where J_2 is the second invariant of the deviatoric stress tensor s_{ij} and α and k are material properties that reflect the friction angle and cohesion, respectively. When f reaches a critical value corresponding to yielding under a uniaxial stress state, the material will yield. Express f in terms of the stresses σ_{ij} that define I_1 and J_2, and determine the value of $f(\alpha, k)$ that corresponds to yielding under an applied uniaxial stress state.

Solution: In Drucker-Prager,

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

and

$$J_2 = \frac{1}{2} S_{\rm ij} S_{\rm ij} = \frac{1}{6} \left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right] + \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2$$

Therefore

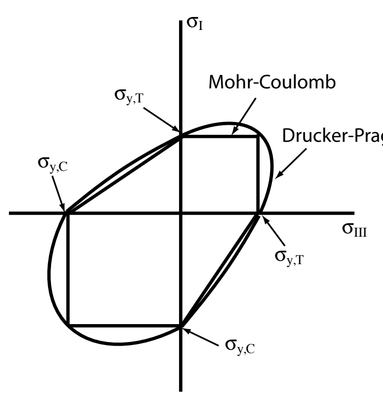
$$f = \frac{1}{\sqrt{6}}\left(\left[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2\right]+\sigma_{12}^2+\sigma_{13}^2+\sigma_{23}^2\right)^{1/2}-\alpha(\sigma_{11}+\sigma_{22}+\sigma_{33})-k$$

In uniaxial tension at yield, $\sigma_{11}=\sigma_1$ *and all else is zero resulting in*

$$\begin{aligned} f &= \frac{1}{\sqrt{6}}[\sigma_1^2+\sigma_1^2]^{1/2}-\alpha(\sigma_1)-K \\ &= \sigma_1(\frac{\sqrt{3}}{3}-\alpha)-k \end{aligned}$$

(c) Mohr-Coulomb and Drucker-Prager are similar in relation to each other as Tresca and von Mises. Both account for superposed normal and shear stresses, but one is more conservative than the other. Graphically represent the M-C and D-P yield locii on the same σ_1/σ_y vs. σ_2/σ_y graph, and state *why* one is more conservative than the other in predicting yielding of pressure-sensitive materials.

Solution: See below for the Mohr-Coulomb and Drucker-Prager yield "surfaces". Because Drucker-Prager consider σ_I, σ_{II}, *and* σ_{III} *(similar to von Mises) and Mohr-Coulomb considers only* σ_I *and* σ_{III} *(similar to Tresca), Drucker-Prager is more conservative in predicting yielding of pressure sensitive materials.*



5. Material stiffness and strength are very different properties attributed to very different atomistic mechanisms. In 60 words or less (think elevator conversation), clearly and accurately state the difference between these two concepts in a way that should be perfectly clear to any engineer, physicist, or chemist.

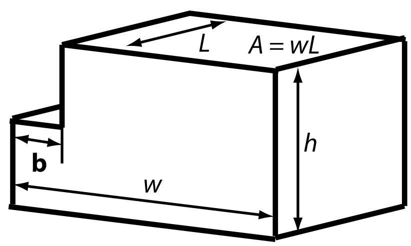
Solution: Stiffness is resistance to reversible bond stretching, and reflects macroscopic resistance to elastic deformation. Strength is resistance to irreversible bond rearrangement and defect motion, and reflects macroscopic resistance to plastic deformation.

6. We have discussed that dislocations move in response to shear stresses less than the theoretical shear strength of a crystal, and that the motion of each dislocation contributes unit slip of magnitude $|\mathbf{b}|$.

   (a) Express the plastic shear strain γ^{p} that would result from a single edge dislocation gliding along its slip plane all the way to the free surface. Here, consider a rectangular single crystal block of height *h*, width across the page *w* and depth into the page *L*; first express $\gamma^{\mathrm{p}}(b, h)$ and then express $\gamma^{p}(A, V)$ where A is the glide plane area and V is the volume of the block.

   Solution: If edge moves entirely thru the crystal on its slip plane then

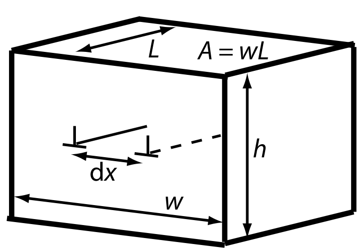
$$\gamma^{\mathrm{p}} = \frac{b}{h} = \frac{bA}{V}$$



   (b) Now consider a single edge dislocation in the center of the perfect rectangular-prism box, gliding by an incremental slip inside the block $\mathrm{d}x$. Express the incremental shear strain $\mathrm{d}\gamma^{\mathrm{p}}$ in terms of $b, L, \mathrm{d}x$ and V.

   Solution: If the dislocation only moves incrementally by $\mathrm{d}x$ *then*

$$\mathrm{d}\gamma^{\mathrm{p}} = \frac{b\mathrm{d}A}{V} = \frac{bL\mathrm{d}x}{V}$$



   (c) Now consider the case of n dislocations, and express this incremental shear strain in terms of dislocation density ρ (which is always defined as total dislocation line length per unit volume of material), b and the incremental slip $\mathrm{d}x$.

Solution: For n dislocations,

$$\mathrm{d}\gamma^{\mathrm{p}} = \frac{bnL\mathrm{d}x}{V}$$

Dislocation density ρ is equal to the line length over volume (nL/V). Therefore,

$$\mathrm{d}\gamma^{\mathrm{p}} = b\rho\mathrm{d}x$$

(d) Finally, express the time derivative of $\mathrm{d}\gamma^{\mathrm{p}}$ in terms of the time derivative of displacement $\mathrm{d}x/\mathrm{d}t$. You will have derived the Orowan equation, which predicts the plastic shear strain rate possible in a material of known dislocation density, Burgers vector, and dislocation velocity **v**.

Solution:

$$\begin{aligned} \frac{\mathrm{d}\gamma^{\mathrm{p}}}{\mathrm{d}t} &= b\rho\frac{\mathrm{d}x}{\mathrm{d}t} \\ \dot{\gamma}^{\mathrm{p}} &= |\mathbf{b}|\rho\mathbf{v} \end{aligned}$$

(e) Assume dislocation velocity is equivalent to the speed of sound in Cu, and determine the plastic shear strain rate magnitude for annealed, single crystal Cu.

Solution: Let sound velocity = $2300\ \mathrm{m\ s^{-1}}$ in Cu [Kumar et al.*; Acta Mat 51: 2417 (2003)] and dislocation density = $10^{13}\ \mathrm{m^{-2}}$ (annealed) and Burgers vector = $3 \times 10^{-10}\ \mathrm{m}$ (these could all be determined more precisely and/or values cited from literature). For these values, plasticity shear rate due to dislocation glide is the product, $7 \times 10^{6}\ \mathrm{m\,s^{-1}}$, or $7 \times 10^{8}\ \mathrm{cm\,s^{-1}}$. Note that this solution assumes that all the dislocations are equally mobile at the same time, and that the dislocations are in fact moving at the shear wave velocity inside the material. In actuality, only a fraction of the total dislocations may be under sufficient stress to move, and the velocity of these dislocations will be less than the shear wave (phonon) velocity; a typical experimental observation in annealed fcc metals is $10^{1}\ \mathrm{cm\,s^{-1}}$.*

7. Complete PS3 wiki questions with your Special Topic colleagues.