PROFESSOR: Do them individually so I can continue to put names and faces together. I'm happy to announce that the registrar has now got everybody's photograph online for registration in the course. So anonymity is a thing of the past, so you have to watch your step from now on.

I handed back, to those who didn't get it, problem set number four, which asked you to tackle some patterns, nontrivial patterns. And actually, that was a dirty trick, because we hadn't, at that point, derived the plane groups, and you really didn't know what to do or what to look for. But nevertheless, it got you thinking about patterns and some of the symmetry elements which we had discussed up to that point.

At this point we have derived exhaustively every last one of the 17 plane groups. So now you are armed with this new-found power, and when faced with a pattern, you should know exactly what to look for and how to go about deciding what plane group it is. At the very least, you'll have the drawings of the arrangement of symmetry elements in the plane groups before you, and you can work by the process of elimination.

For example, high symmetry usually hits you right between the eyes, and if something is square-ish, you can pretty quickly guess that it's based on a square lattice. And if it has a square lattice, there jolly well better be a 4-fold axis in there that makes it square. If you can find the 4-fold axis, then you have to ask yourself only three questions.

So 4-fold axis, fine. Is there a mirror line in there? Yeah. Does the mirror line go through the 4-fold axis? Then it is P 4 MM. And you know just where to look for everything else, including these very subtle glide planes that are hard to spot. If there is a mirror plane there, but it doesn't go through the 4-fold axis, then it's P 4 MG. And if there is no mirror plane, then it's P 4.

So just by asking one or two simple questions, you can narrow it down to what the plane group has to be. This is another indication that the informed intellect is always more than a match for sheer, raw native intelligence. If you know what to look for, it's a lot easier.

Because you really didn't have much practice with patterns, we're having a quiz, as you know, next Thursday, that will cover up through completion of the plane groups and not the material we've been doing now. So I think it might be of use to you to have some practice analyzing a few more patterns.

So there are four additional patterns in this problem set. As always, it's optional, but if you would like to try them, and you want to see if you've got them right, come in and see me tomorrow or on Thursday morning. I'd be happy to go over them with you on the spot.

So this is for practice. And some of you did extraordinarily well on the first try. Others, I think, could use the additional practice. There were a few people who identified the plane group correctly, but got the name that's assigned to it wrong. And one other very confusing thing is that there is one plane group that has the symbols G and M, and another plane group which has the symbols M and G.

So if you found a mirror plane and a glide plane is an independent symmetry plane, when is it MG, and when is it GM? I have a little mnemonic device. GM, general manager, is the guy who sits on top of the organization. So GM should be the plane group that has the highest symmetry. P 4 GM, P 4 general manager.

Now that is-- hey, it works for me. But another way of saying it is that there are two plane groups, one has MG and the other has GM, and I like cars, and I think an MG is much classier than anything that General Motors, GM, puts out, so MG should be the one of highest quality, highest symmetry. And that's just the reverse.

But whatever works for you. You could keep them straight through that simple algorithm. And as they say, if it works for me, but if it doesn't work for you, don't use it. All right. What I will bring in during intermission, for those of you had trouble identifying translations in the patterns that we handed out earlier, I've taken these and put them on overhead transparencies. And I'll have two of each.

So if you don't see the symmetry or translations that are present, you can actually take one pattern and physically move it and lay it on top of the other one, and that's a good way to convince yourself what a translation looks like when it occurs in a pattern. So I'll bring those in at our break between class. All right, any questions before we move on?

Any questions that have arisen as you have gotten ready for the quiz? You haven't gotten ready for the quiz yet, so there are no questions. That's OK. I know how things work at MIT. You deal with one crisis at a time. Any questions? Anything you want to go over?

There was one interesting wrinkle in a problem that I had not encountered before, and this was the one that asked you to look at, in two dimensions, a plane with indices h and k. And then, when h and k were mutually prime, to move that plane by the translations plus T1 and plus and minus T2, and then show that the number of intervals between the origin and the intercept plane, the one that hit lattice points on both translations, was equal to h times k if they were mutually prime.

That is true only if the lattice is primitive. And what the problem said was to pick one of the cells that you used in the first problem, number one. Well, that problem asked you to begin with identifying different primitive cells. If you take a multiple cell, this operation of going plus and minus T1 does not put a lattice line through each of the lattice points.

If you picked a double cell, that process decorated only half of the lattice points with planes, and the other half sat there with nothing hanging on them at all. The key to the difference, if you looked at a double cell, was that if h plus k was even, then you automatically got a plane on every lattice point.

If h plus k was odd, as it would have been for the plane 2,1 for example, 2 plus 1 is 3-- hey, this isn't even one of my good days-- then half of the lattice points did not have planes hanging on them. Now, there's great relevance of this observation to diffraction, and you probably are all familiar, if only vaguely, with the magic rules that say, if h plus k is equal to 3 pi plus 4, then the intensity is identically 0.

Well, for a double cell, the lattice planes that are repeated by translation diffract x-rays, and there is no reason why the intensity should be something other than 0. So here comes, a la Bragg, an x-ray beam coming in at angle theta, and then you say you get diffraction when scattering from this lattice plane is exactly in phase with this one.

And this gives the familiar relation that an integral number of wavelengths is equal to 2 d sine of theta when the crystal diffracts. So there is exactly-- this says there's exactly 2 pi phase difference or n lambda path difference between these two planes. Now, if the lattice would be a double cell, then there is an additional lattice point in here that does not get a plane hung on it.

So if the lattice is a double cell, there's another plane that has to hang on this lattice point, and that one is exactly out of phase with this plane, and the intensity is 0. So this observation that a non-primitive lattice has a interplanar spacing that is a sub-multiple of that of a primitive lattice gives some insight into why certain reflections-- certain diffraction maxima-- are identically 0 in intensity for a crystal that has a non-primitive lattice.

So that is something I had not noticed before. I should have, but I will phrase the problem a little bit more precisely in the future. All right. So to conclude my preamble, I hope you'll try playing with some of the four additional patterns that I handed out, just to give yourself some practice.

And the implication of this is that you're going to see a pattern on the quiz, and I will tell you that you will. So if you want to see how you did on the patterns that I distributed, please come in and talk to me about them.

All right then. Let me remind you where we were last time. We started to begin to build a framework of symmetry elements in three dimensions. And we asked the question, what would happen if we take a first rotation axis, A alpha, combine it with a second rotation axis, B beta, in such a way that they intersect at a point.

This means that their operation and reproducing atoms or motifs is going to leave at least one point in space unchanged, and that will be the point of intersection. We ask ourselves, what will be the combined effect-- we have two operations in space-- what would be the combined effect of rotating alpha degrees about A followed immediately by beta degrees about B.

So what we're going to do then is to take a first motif-- and let's say it's left handed. Being a left-handed person, I like to give right handed motifs and left handed motifs equal time. If we rotate that through an angle alpha, and this is number 2, it will stay left handed. Then if we rotate that by B beta, it'll move it over here to number 3 and it will stay left handed as well.

And the question is then, what net operation is equivalent to the combined operation of these two transformations? And to specify the type of operation is really a no-brainer. All of these motifs are of the same corality so the only thing that can relate them is translation or another rotation.

And clearly the first and the third have no reason to be parallel to one another, and the distance between them is going to depend on how far they are away from the axis, so translation won't do the job, a and the only thing that's left as a net operation that's equivalent to those two steps is rotation about a third axis C.

And what we're going to do today is answer the question, where is axis C located, and what is the angle of rotation, given the value of alpha and beta and the angle between these two axes? And let's define that as a lowercase c. So clearly the location of the axis and the amount of the rotation is going to be a function of alpha, beta, and the angle between them.

We want this to be a combination of operations that exists in a symmetry operation. And if this is to be a crystallographic symmetry, these will be restricted to the angular throws of a 1-fold, 2-fold, 3-fold, 4-fold, a 6-fold axis. We can take these two at a time, ask what the net effect is if we combine at a given angle c.

And what comes out here must be a rotation which is also crystallographic. So there are going to be severe constraints on this combination. Two rotations about an intersecting point will always be a third rotation, but if this is to be a set of operations in a symmetry group, the result must be crystallographic.

That's a tough problem, and how will we undertake it is going to be non-intuitive. OK, the problem is most readily treated with spherical trigonometry. So on the surface of a sphere, I'm going to map the point at which A alpha protrudes-- and I'll call this point A-- and then I'll mark out the point where axis B beta exits the sphere, and I'll mark that point B. And this is the angle C.

And we said that in spherical trigonometry, the measure of the length of the arc separating A and B is given by the angle subtended at the center, so the length of this distance between A and B is the angle c. Again, it sort of boggles the mind when you measure lengths in terms of degrees rather than some metric unit.

All right. So I will now not bother to show the sphere on which the geometry is taking place. I'll just draw A and B, and this is the arc between them, c. And somewhere or other there will be some third axis, C, which is going to be the combined effect of the rotation about axis A and axis B. So what I would like to do is to locate the position of this axis C.

In order to do that, I'll have to know what the angle between A and C is, and I'll call that, by analogy to what I've done here, I'll call that b. And I'll want to know what the angle between B and C is, and I'll call that angle a. So again, going back to three dimensions momentarily, if this is the rotation operation C gamma, and this is A alpha, and this is B beta, the axis c is this, the angle b is this, and the angle a is this.

OK. It's a non-trivial problem and it is not by accident that the solution to this problem was first given by a very, very famous mathematician, Leonhard Euler, and this construction that we're about to go through is called Euler's construction. All right. Let me find where these different locations are going to be.

We've specified the location of point A and the location of point B, and we know that the angle between them is the length of the arc ab, which is the angle between A and C. So let me now do some constructions. Let me find a great circle that by design is alpha over 2 away from the arc ab, and that is by construction.

And I'm going to say, then, that if A alpha works in this direction, the operation of A alpha is going to take this great circle and move it over to a great circle which is alpha over 2 on the other side of the arc ab. Fine, you say, so what? Well, just going to leave those there for now. I'll have B beta work in the same sense.

And I'm now going to create a line here that by construction is beta over 2 on one side of the arc ab, and if I let B beta go to work, that will map this great circle over to a new location beta over 2 on the other side. What has this done for me, other than perhaps confuse me and clutter the diagram?

Well, now I'm going to determine unequivocally the location of the axis C, and where it emerges from the reference here. And how will I do that? I'm going to use a definition that may have seemed trivial the first time we made the observation. I said that a symmetry element is the locus of points that is left unmoved by an operation. OK?

I rotated by A alpha from here to here, that took everything along this line and mapped it to a new location here. I took this line and rotated it by B beta, and that took everything along this line and moved it to a new location. So my question now is if I rotate by A alpha and then rotate in the same direction by B beta, what point is left unmoved?

It's only one point that can make that claim, and that is where these two great circles intersect. The rotation A alpha will take this location-- and I'm going to call it C because I've identified now what it is-- it's going to take C and move it over to here, call that C prime, and then B beta takes that point and only that point, and restores it back to its original location.

So this, then, ladies and gentlemen, is where the rotation axis C gamma pokes out of the sphere of reflection. Still don't know what this angle is in here, and I would dearly love to know what these arcs b and a are, and then I will have specified all three of the interaxial angles between A, B and C.

OK, let me do something quite similar to what I did before. I'm going to again let A alpha work on a particular point, and then let B beta map it. So here's A alpha, here's B beta. And now I'm going to look specifically at how these two rotations transform point A, where A is the point at which axis A alpha pokes out of the sphere.

A alpha, when it acts on this point, does nothing to it. It leaves it alone. B beta is going to map A to a new location, A prime. Now, doing A alpha and following up by B beta is supposed to be equal to the rotation C gamma. So that says that this point and this point must be related by the rotation gamma.

So say that again. We're doing exactly what we did here except we're starting with an initial point, not this arc, but we're starting with the specific point A, operate on it by A alpha, it twirled around but stays put. Rotate that by B beta, it goes through a total angle beta to this location here.

The net effect of getting from A to A prime is supposed to be the rotation C gamma, so this angle is then gamma and this is the location of C. OK? OK, one other step that's a fairly easy one. This length is equal to this length, because they were produced by rotation. This side is common to these two triangles, and this angle then is beta over 2, this is beta over 2.

And if these two triangles, A, B, and C, that triangle is similar to A prime BC, and therefore I can say that angle A prime CA is identical to ACB, so therefore this angle has to equal this angle, and if the total angle is gamma, this is gamma over 2, and this is gamma over 2.

So now let me extract from this the information that I would like to use. Here are three axes, A alpha, B beta, and C gamma. This angle in here is gamma over 2. This angle in here is beta over 2, and this angle in here is alpha over 2.

And let me emphasize that in this magic triangle, out of which we're going to extract some dazzlingly profound stuff, it is half the angular throw of the rotation axes that appear in here as these spherical angles, and not the entire angle of rotation. So here's how properties of the three rotation axes are related one to another.

And now, we introduced without proof last time something called the law of cosines in spherical trigonometry. And I not only do not want to prove it, but I have no idea how I would go about doing so, but that doesn't prevent me from using it.

So if here are three edges, a, b, c, and three angles in there, A, B, and C, we said that the law of cosines in spherical trigonometry, analogous in a way to the law of cosines and plane geometry, except since the lengths of the triangles are measured in degrees, there are trigonometric functions of these angles that appear in the law of cosines.

This says that cosine of b cosine of c plus sine b sine of c times cosine of a is equal to cosine of a. So this now is an interesting relation that we can apply to this spherical triangle, which connects together the three rotation axes. Let me apply it to find the angle c which we have picked as the angle between the initial two axes a and b.

That says that this should be equal to cosine of b cosine of c, the angle between the other two axes, plus sine of b sine of c times the cosine of angle a, and angle a is cosine of alpha over 2. So all these quantities that we'd like to determine are hooked together by the law of cosines.

And this is a lovely relation, but it doesn't do us a bit of good, because in this relation we know only one quantity, and that is the rotation angle of a. We can pick the angle between a and b, that's this, but I have no idea what these other angles are. That's what I'd like to find out. I'd like to find out the angles at which three rotations have to be combined in order that rotation about one followed by rotation about the second be the third.

So this equation is a beautiful equation, but it involves everything that I don't know and only one quantity that I do know. So it looks as though we're up the creek. Yes, sir?

AUDIENCE: So you're looking for cosine c, so shouldn't it be cosine b cosine a?

PROFESSOR: Oh, I'm sorry. Yeah, I did that wrong. Yeah. You're absolutely right. This should be cosine of a, and this a goes with this alpha over 2. Absolutely. Sorry about that. So anyway, the point still stands that what this equation involves is the three interaxial angles, and I would like to know how I could combine a and b to get it to come out to a crystallographic rotation c, and where that location is relative to the first two axes.

So it involves everything I don't know, and only one thing that I do. But now we introduce another curious aspect of spherical triangles, which I mentioned last time. You may have thought that that's interesting, but who cares? Here are the three points, A, B, and C, and these are the three arcs little c, little a and little b.

And then we said we could construct something called the polar triangle of ABC. And what we would do, we would find the pole of arc b, and that will be some point B prime. We'll find the pole of arc a, and that will be some point A prime. We'll find, similarly, the pole of arc c, and that will be some point C prime.

And now we can connect together A prime, B prime, and C prime, and get something that's called the polar triangle. And now comes the useful part. We said that a curious property of the polar triangle is that the side of the polar triangle plus the angle opposite it add up to 180 degrees.

In my original triangle, this is beta over 2, this is gamma over 2, and this is alpha over 2. So the length of this side is going to be 180 degrees minus beta over 2, the length of this side is going to be 180 degrees minus alpha over 2, and the length of this side is going to be 180 degrees minus gamma over 2.

And now let's use these angles and these lengths in the law of cosines, and I'll leave out the little bit of intervening algebra. And what we will get out of this is that cosine of c-- and I'll solve for that-- is equal to cosine of alpha over 2 cosine of beta over 2 plus cosine of gamma over 2 divided by sine alpha over 2 sine beta over 2.

And that is something we can sink our teeth into and run with, because now I can ask the question, suppose I want a to be a 2-fold rotation axis, b to be a 3-fold rotation axis, and c be a 4-fold rotation axis? Then the value of alpha over 2 is half of 180 degrees or 90. Well, you can see I put in half the value of the rotation axes.

And then I solve for c, and that is the angle at which I have to put axis a and b together to get c to turn out to be whatever angle gamma over 2 is. So I can do this systematically now without thinking. And I can set up the problem by taking the crystallographic rotation axes and combining them together three at a time in all possible combinations. Right?

In addition to this relation, I have two other relations. And let me assemble them off to the left, because we have to solve three equations to find out the nature of the combination that is required. So just permuting terms, the angle between A and B, c, has to follow from cosine of c equals cosine of alpha over 2 cosine of beta over 2 plus cosine of gamma over 2.

Notice that the single term by itself is the cosine of half the angle of the opposite rotation axis c. Then in the denominator is the sine of these two angles. And so just permuting terms, one can see that cosine of b is going to turn out to be cosine of alpha over 2 cosine of gamma over 2 plus cosine of theta over 2 divided by sine of alpha over 2 sine of gamma over 2.

And a third analogous expression will give me the angle that will be the one between B and C. And this will be cosine of beta over 2 cosine gamma over 2 plus cosine of alpha over 2 divided by sine beta over 2 sine gamma over 2. OK? So now we don't have to think anymore. It's just plug and chug.

And I'll pause to suck in air and let you catch up, and then we'll set up the problem and look at a few solutions. And all this, in the event that you're thoroughly bewildered, is in the set of notes that I handed out last time. So you can read it over at your leisure.

AUDIENCE: Will this stuff be on the quiz?

PROFESSOR: No. Quiz will go up to the end of the two-dimensional plane groups and stop. We won't say anything three dimensional. OK, let's, then, if there's no objection or complaint, look at possible values for-- let me do it the same way that I did it in the notes so that it's consistent-- let's put down the value for axis b, the rank of axis b and the rank of axis a.

And A could be a 1-fold axis, B could be a 1-fold access, and we could take a 1 with a 1 with a 1, a 1 with a 1 with a 2, a 1 with a 1 with a 3, a 1 with a 1 with a 4, and a 1 with a 1 with a 6. This is clearly impossible. If I did nothing, and followed it by doing nothing, and wanted it to come out to be a 6-fold rotation, you'd all be spinning on your axes like tops right now.

So you can't do nothing and follow it by doing nothing and have it come out to be a net rotation. So these are impossible. So we don't have to consider those. A could be a 2, though, and I don't want to do 2, 1, 1, because I've got a 1, 1, 2 here. The order doesn't make any difference. So I'll start with a 2, 1, 2, a 2, 1, 3, a 2, 1, 4, and a 2, 1, 6. So those are four combinations that I should be examining.

I could look at a 3 with a-- 2 with a 1, I have here in the form of 2, 1, 3, so the next one I would want to look at is a 3, 1, 3, a 3, 1, 4, and a 3, 1, 6. And let me put in a couple more here. If B were a 2, I should look at a 2 with a 2 with a 2, a 2 with a 2 with a 3, a 2 with a 2 with a 4, 2 with a 2 with a 6.

And then A 3 with a 2-- and I've got 3, 2, 2 up here, so I'll start with 3, 2, 3, 3, 2, 4, 3, 2, 6, run out of room here, but there should be a similar entry with a 4 and a 6. So this sets up the problem. If you count up the number of ways one can do this, we only have to consider the off-diagonal boxes here, because interchanging a and b, for example, looking at 3, 1, 3, that's going to be the same as 1, 3, 3 up here.

So it's just the off-diagonal boxes that we have to consider. So there should be a 3, 3, 3 in here, a 3, 3, 4, and a 3, 3, 6. So what we would have to do in order to determine the unique combinations is to look at all of these combinations in turn, and I'm going to not try all of them.

I will do one that's going to be clearly impossible. So let's look at 2, 1, 4. So here A corresponds to a 2-fold axis, B corresponds to a 1-fold axis, and C would correspond to a 4-fold axis. So could we combine these three axes at appropriate angles such that a 2-fold followed by a 1-fold is equivalent to a 4-fold?

This clearly isn't going to work. If I do a 180-degree rotation then don't do anything and ask is that equivalent to a 4-fold rotation, that is saying that the 2-fold axis should be identical to the 4-fold axis, and that is not going to work. So let me do now a generic family that I know turns out to be possible.

Let me look at an n-fold axis with a 2-fold axis with a 2-fold axis, and I can show that this combination is possible for any integer n whatsoever. So this will include a lot of non-crystallographic symmetries. So let's say that this is C, this is A, and this is B. But make it C, B, A if you'd like.

OK, so my first equation says that cosine of c, the angle between A and B, should be equal to the cosine of alpha over 2 times the cosine of beta over 2 plus the cosine of gamma over 2 divided by sine of alpha over 2 sine of beta over 2. B is a 2-fold axis, so alpha is equal to 180 degrees. A is a 2-fold--

I'm sorry. B beta, A alpha. Beta is equal to 180 degrees, the angular throw of a 2-fold axis. Alpha is equal to 180 degrees, the angular throw of a 2-fold axis. And gamma is equal to whatever 2 pi over n would be. That's the throw of the n-fold axis than I'm letting be equal to C.

So the cosine of A and B, to get the result of rotation A followed by rotation B being equal to the net rotation of an n-fold axis is that the cosine of c should be the cosine of 180 degrees over 2 times the cosine of 180 over 2 plus the cosine of gamma over 2, and gamma is whatever the rank of the axis determines, and that's divided by sine of alpha over 2, and alpha is 180 and sine beta over 2, and that's 180 over 2.

So the cosine of c is going to be the cosine of 90, which is 0, times the cosine of 90, which is 0, plus the cosine of gamma over 2, whatever that might be, over the sine of 90 which is 1, sine of 90 which is 1. So this says that cosine of c is equal to cosine of gamma over 2.

So the angle between axis A and axis B ought to be equal to one half the angular throw of rotation axis C. So let's start putting down some of this information. This says that if this is axis C gamma, then A pi and B pi, the 2 180-degree rotations, should be at an angle gamma over 2.

We still need values for b, and we still need a value for a. So let's find out what those are. And let me start over here at the left again, because I've got the relation that I need for b sitting here. The cosine of b is equal to the cosine of alpha over 2, and that is the cosine of pi over 2, plus alpha is a 90-degree rotation.

Then cosine of gamma over 2, whatever gamma happens to be, plus the cosine of beta over 2, and that's cosine of pi over 2, and this is all over sine of pi over 2 times the sine of gamma over 2. So this is going to be cosine of pi over 2, which is 0, plus cosine of pi over 2 divided by sine of pi over 2, which is 1.

This gamma over 2-- no, cosine of pi over 2 is 0. So this is 0 plus 0 times sine of gamma over 2, whatever that turns out to be. So cosine of b is 0, and that says that the angle b between axis A and axis C turns out to be 90 degrees. So in order to get pi followed by c gamma to be equal to b pi, I've got to make b be equal to pi over 2.

And if I put it in this orientation, then a pi followed by c gamma is going to be equal to b pi. One final angle, and that's the value for a. OK, cosine of a should be equal to the cosine of beta over 2, that's pi over 2 times the cosine of gamma over 2, whatever it is, plus the cosine of beta over 2, and that's cosine of pi over 2, over sine pi over 2 sine of gamma over 2.

And this, as for b, turns out to be 0 plus 0 over sine pi over 2, which is 1 times sine of gamma over 2. So cosine of a turns out to be 0, and this says that the angle a is also pi over 2. So we've got a whole-- Yeah.

AUDIENCE: Did you really need to go through all three equations?

PROFESSOR: Yeah, because I had to show that all three work. And in general, if I combine, let's say, a 4-fold with a 3-fold with a 2-fold, which is something I want to do, all three angles a, b, and c, will be different. OK? So the answer is yes. And there will be a few cases where a value for one angle will exist and the value for the one or two others will be impossible. And that's also something that I have to know.

So repetitious as the exercise might be, the answer is yeah, you do have to do all three.

AUDIENCE: [INAUDIBLE]?

PROFESSOR: Oh, you don't have to do different permutations. That's just a question of labeling. OK? So n with a 2 with a 2 is the same as a 2 with an n with a 2 is the same as a 2 with a 2 with an n, that's just labeling. So that's why my boxes, when I filled them out, the list got shorter and shorter, until finally for a 6-fold axis, it would be just 6, 1, 1, 6, 1, 2, 6, 1, 3, and so on. Just that one entry in the box where I enumerated what should be considered.

Well, here is a whole slew of possible solutions and a lot of them are non-crystallographic, but still possible. This says that a combination of a 2-fold axis-- but remember now that these are equations and operations. But the only operation that's present for a 2-fold axis is a rotation a pi, b pi or c pi.

So I could combine three 2-fold axes that are mutually orthogonal, and that is an allowable combination. And what we are obtaining here is a sort of scaffolding, a framework, based on pure rotation operations, that by themselves will be an allowable 3-dimensional point group, but which also provides a framework, a Christmas tree, that we can decorate with mirror planes and inversion centers to get still additional groups of higher symmetry.

So here's one possible crystallographic o combination of rotation axes. What we will use to denote this combination is the same rule that we use for our other notation. We will make a running list of the independent operations that are present, and what we have combined here are three distinct independent 2-fold axes.

A solid that would have this symmetry plus some other symmetry would be an orthogonal brick with one 2-fold axis coming out here. This is the operation c pi, another 2-fold axis coming out the front, and this would be the operation a pi, and another 2-fold axis coming out of this face, and this would be the operation b pi.

Now let me show you-- it's rather amusing-- that what we have done really works. We've shown supposedly that a pi followed by b pi should be equal to a net rotation c pi about an axis that's orthogonal to the first two. So let's pick a motif, and for convenience I'll put it at one corner of this brick.

Here's object 1, I rotate it by 180 degrees about a. Here sits object number 2, same corality, and then I rotate it 180 degrees about B beta, and that's going to give me number 3. What is the net way of getting from 1 to 3? Holy mackerel. It's a net 180-degree rotation about c pi. It really works.

Or I could do the operations in a different order. I could rotate by d, rotate by c, and the way I get from the first to the third is a rotation a pi. So that is a self consistent set of rotation axes. That is 2, 2, 2. Let me do one more. Well, no, let me take a break here and let you absorb all this, and then we'll look at some remaining ones, and this will include some that are non- crystallographic.

And that's perfectly OK. But they're lovely groups, they constitute groups, but they won't be groups that can occur in combination with a lattice.