PROFESSOR: We had begun to discuss, really what is, although you would not gather it from last time, a fairly straightforward matter. Set up our usual Cartesian coordinate system. And then look at the surface of the solid, and applied to this solid a force per unit area, which we'll represent by a vector k. And then put on some labels here. Let this be O. Let this be A, let this be B, and this be point C. Then we proceeded to-- I will say, attempt to-- set up a balance of forces between the external force per unit area and the internal forces per unit area, which have to push back on the external force if the solid is to remain motionless.

The thing that I did not carry through is that there are two directions involved. One is the direction to the normal, to the particular surface that we're looking at. We'll call that N. And there is the direction that k, the vector k, makes with respect to the three reference x axes, x1, x2, x3. Let's take k and immediately split it up into the three components, k1, k2, k3, because throughout, we're going to be dealing with the balance of those three forces per unit area in the directions of x1, x2, and x3.

We want to take this surface area and resolve it into three separate areas that are normal to x1, x2, x3. And what I did at the end of last hour was to look at k and try to express these internal areas in terms of the direction cosines of k. Forget that. We've split up k into its three components, and I didn't draw in the direction of the normal to the surface N last time. So I kept saying, direction cosines of k, and people in the audience said, no, no, no, no. And I said, yeah, because you see, that's the ratio of these two areas. And everybody said, no, that's wrong, and I said, ha. And that's where we ended my most inglorious hour. And they've got this all down on tape. My reputation's going to be ruined.

So, anyway, let's now do it properly. And what I will say is that in the x1 direction, we have to have internal forces that push back, and we have to balance those forces in all three directions. So I'll say is that in the x1 direction, and this as you saw last time, gets very cumbersome because we have to deal with all these clumsy areas, that the force per unit area on ABC has to be balanced for each of the three components of k by internal forces acting along x1, x2, and x3. So we can say is that the first in the x1 direction, the component of k that acts on the surface ABC in the x1 direction has to be balanced by internal forces.

And let's look first at the x1 direction. So there are three internal surfaces. They are the area BOC. That is this back surface here, and the normal to that surface is x1. So let me define some forces per unit area that are internal, and I'll use the term sigma for that. And I will use a first subscript that gives the direction in which that force per unit area acts, and I'll use a second subscript that indicates the normal to the internal surface.

So area BOC obviously, as we observed a moment ago, has x1 as its normal. But there are three internal surfaces. There will also be an internal area, AOC, on which a force per unit area acts in the x1 direction. So I'll call this one by analogy to what I did a moment ago. I'll call this sigma 1, because it acts in the x1 direction, and it acts on the surface AOC, which has x2 as its normal.

Then finally, there will be a third force per unit area, sigma 1 3, acting in the x1 direction, on a surface whose normal is perpendicular to-- along x3. The surface is perpendicular to x3. And that's area AOB. OK, so this is just a simple balance of forces, external and internal balancing forces, which act in the x1 direction.

Then we'll write a similar thing for the x2 direction. And we'll say that the x2 component of force per unit area external to the volume element acting on area ABC is going to be balanced by, again, forces now acting in the x2 direction. So first, there will be a component of 4 sigma 2 acting on the area that's normal to x1, and that's the same area BOC, plus another force per unit area acting in the x2 direction on the surface, whose normal is a long x2. And that is, once more, area AOC. And that would be a force per unit area that I'll define, because it's my ball game and I own the ball. So again, that would be sigma 2 2, and then finally a sigma 2 3 times the area whose normal is x3.

And I think I need not write the third term that follows automatically from what I've written so far.

So now comes the geometry part, where I'm going to get rid of these messy areas, and I'm going to divide through by the external area on the surface of the solid. I will then write accordingly that k1 is equal to sigma 11 times the ratio of area BOC divided by area ABC plus sigma 12 times area AOC divided by area AOB-- ABC, excuse me. Then finally sigma 1 3 times area AOB divided by area ABC.

So now I'll introduce a little bit of geometry that caused such difficulties last time. If I have an area and project it onto another area that makes an angle C with respect to the first, this area A prime is equal to the area A times the cosine of Phi and you could define that Phi in a different way, namely take the normal to one of the two areas and let M prime be the other normal, so C then is exactly the same thing as the angle between the normals to these two surfaces.

So let's now look at what the ratio of these two areas that we have is. What we're dealing with here is an angle between the normal to the surface, not k, but the normal to the surface, and either x1, x2, or x3. So the ratio of-- let's look at one of these specific terms. The ratio of area BOC through the ratio of the area ABC should be the cosine of the angle between the normal to the surface and the normal to area BOC.

Well, the normal to area BOC is x1, so that first ratio of areas is just going to be the cosine of this angle. And that cosine is just the direction cosine of N1 along the x1 of M along the x1 direction. So this is just the direction cosine between N, the normal to the surface, and x1. And let me say then that, by definition, I'll let the direction to the normal be its three direction cosines, L 1 plus L 2, plus J plus L3 times k. These are the direction cosines of the normal to the surface, and not k.

So this then simply comes down to sigma 11 times L1, the ratio of the area AOC to ABC is going to be the angle between N and x2. And that's the angle whose direction cosine is L2. And similarly, the ratio of these two areas will be cosine minus 1 of L3.

So this cumbersome construction with ratios of areas just reduces to sigma 11 L1 plus sigma 12 times L2 plus sigma 13 times L3. And the other two equations will follow a similar form. K2 will be sigma 21 times L1 plus sigma 22 times L2 plus sigma 23 times L3. Or in general, case of I will be given by sigma I J times L to J. Easy. Nice and compact. Nice and simple.

What can we say about this? K is a vector which has components k1, k2, and k3. The direction cosines can be, as we said a moment ago, can be regarded as the components of a unit vector pointing along the normal to the surface. So k is a vector. These three direction cosines are the direction cosine to a unit vector normal to the surface. If we were to be so foolish to try to change the coordinate system., having just gotten it right for the first time in one coordinate system, we know exactly how the two vectors will change because we have a law for transformation of a vector. If we were to change the coordinate system, the components of k would wink on end of and take new values and we know exactly how the components of a vector will change as we change coordinate system.

OK, if this transforms like a vector and this transforms like the vector, then the 3 by 3 array of coefficients which relate those to quantities must be a tensor. So we've proven rigorously that sigma IJ is a tensor, in particular, it's a second rank tensor. And were we to change coordinate system, we know from what we've done in the past, exactly how the numbers that make up this 3 by 3 array will transform.

These quantities, sigma IJ, are something that you've probably encountered, introduced to be sure, probably in a little different way. These are the elements of stress, and sigma IJ is the stress tensor. We can immediately, on physical grounds, established that sigma IJ must be a symmetric tensor. So let's set up two axes, x1, and x2, and x3 obviously is normal to the board.

And let us look at some off-diagonal terms like sigma 12 and sigma 21. Sigma 1 2 would be a force acting on the x1 direction on a surface whose normal is x2. So this is going to be a sheer stress and that, to prevent any translation of the volume element, must be balanced by a sigma 12 on the other face that acts in the opposite direction.

We're going to need a convention for defining sign, S-I-G-N, of what constitutes a positive element of stress and a negative element of stress. And I'll define a plus sigma IJ as a force per unit area acting in the plus xi direction on a surface whose normal is plus xj.

So if this is a force per unit area acting on this surface, and it acts in the direction of x1 on a surface whose normal is x2, what I've shown here is a positive sigma 12. A sheer force going in the opposite direction would be a negative sigma 12. A positive sigma 11 would be a stress that's tensile, a negative sigma 11 would be a stress that's compressive. So here's a sigma 12, a sigma 12 on the opposite face. If we went away and turned our back, this volume element would start accelerating, getting an angular acceleration, would start furiously spinning about the x3 axis like a top. So something, if this volume element is, by definition, an equilibrium, something has to balance this couple. And if it's to stay put, we must have a sheer force like this and a sheer force like this. And that's the only thing available from the nine elements of stress that will give a net torque on the volume element.

So what are these? This is a stress that acts on the surface whose normal is x2-- whose normal as x1 in the x2 direction. So this is sigma 21. It acts in x2 direction, and its normal is x1. An x, and then [? plus ?] x2 direction on [? I ?] surface, which has a normal that is along x1.

So the only way we can avoid an angular acceleration is to say that sigma 12 has to be identical to sigma 21 if there's no rotation about x3. And looking at the couple's that act about x2, and x3, we could say in general that sigma IJ is going to have to be equal to sigma JI for a volume element in equilibrium. So stress then is equilibrium is in place, has to be a symmetric tensor.

It is, however, a plain old second-rank tensor like all of the other second-rank tensors that we've spent the last month talking about. And I think I tried to swindle you on this one before. If this is a second-rank tensor, the stress tensor must be symmetric and diagonal for a cubic crystal. In other words, the stress tensor sigma IJ must have the form sigma 0 0, 0 sigma 0, 0 0 sigma because we've shown that

that is what is required of a tensor if the crystal has cubic symmetry. So surprisingly, you cannot subject a single crystal with cubic material to sheer stress.

Any quarrel with that? Should we move on? We better not, but somebody better raise an objection. That's not true, obviously. I can take a cubic crystal and shear the stuffings out of it. So what's going on here?

AUDIENCE: The tensor property that you are applying to [INAUDIBLE] property of the [? crystal. ?] But if it is an outside force, it doesn't [INAUDIBLE].

PROFESSOR: This is one of your good days.

AUDIENCE: [INAUDIBLE] society [INAUDIBLE].

PROFESSOR: That makes it an especially good day because you're paying attention to what I said for a change.

The distinction here, and I did make it last time-- it was one of the things that I did correctly-- that there are really two kinds of things that have to be described by second-rank tensors. There are the things that we have been spending most of our time on, and these are property tensors and-- I mentioned last time, too, so I know I said this before-- Nye calls them field tensors. I'm sorry. Nye calls them property tensors, also. And these are something that depends on the structure and bonding and other characteristics of the material.

The stress is something that we we're taking and imposing on an innocent, unsuspecting crystal, and that is something we do. And we can, by design, twist it, shear it, stretch it, as hard as we like, provided we don't break it. And so this is something that is externally imposed on the crystal, and this is something that's called a field tensor. We mentioned last time the meaning of field. The field is a volume of space and a field tensor is a space in which a value of the tensor is defined as every point, x1, x2, x3, in the space. Just as an electric field is a vector field and you define the electric field we as a function of every point x1, x2, x3, in the space.

So there's no reason whatsoever that there need be restrictions on the tensor that we apply. And when we can cross out that interesting, but incorrect, supposition. But they can be a lot of different forms, special forms, of stress tensors which have special forms simply because of the nature of what we do. And these have restrictions and equalities that have no counterpart in property tensors. For example, we can have the cancer sigma 00000000, which is something we could never have identically so for a property tensor.

And what does this represent? This is a tensor for which sigma IJ is 0, unless I equals J equals 1. So this is the only thing that's going on, is a force per unit area along the x1 direction. So this is a uniaxial stress. And that is a stress field which we could apply on a piece of material very easily. Just screw a hook into the material and hang a weight on the hook and you've got uniaxial stress.

We can also have something that's a little harder to visualize, a biaxial stress. And this would be a stress that had elements sigma 11, register 0-- at least when we refer to principal axes-- 0 sigma 2 2 0 0 0 0. When it's in diagonal form, two of the diagonal terms are 0. That doesn't seem to make sense because, if this is a force per unit area along x1, and this is a force per unit area along x2, can't we add up those vectors and say that that's a uniaxial stress? Depends on your coordinate system.

If you want an example of a biaxial stress, imagine some firemen underneath a burning building, and they're holding out a blanket for somebody to jump into. You have one pair of firemen pulling in this direction to keep the blanket taut, and another pair of firemen are not quite so vigorous, and they're pulling in an orthogonal direction. And those two forces, sigma 11 and sigma 22, create a biaxial stress event material. You could change the form of the tensor by rotating the coordinate system, but this is one example of a biaxial stress.

The final general form, a triaxial stress, obviously by extension, is going to be where of all three diagonals, sigma 11, sigma 22, and sigma 33, are non-zero when you put the tensor in diagonal form. This would be a general triaxial stress. And an example of that would be when the jumping person lands on the blanket, there's going to be a force that his or her mass exerts on the blanket, and that would be a third sigma 3 3. So that would be a triaxial stress.

Let me give you a couple of special cases of specialized stress tensors. And suppose I had a triaxial stress, but all the diagonal terms were equal. That's something that I tried a moment ago to hoodwink you into believing was required by a cubic crystal, but suppose that was the form of the stress tensor. What would that represent?

AUDIENCE: [INAUDIBLE]

PROFESSOR: That is something that's going to be-- yeah, very good. That's not a pressure. It's a dilation. But if I make it into a more familiar form, put it here, minus P, minus P, minus P. It's a force prevented area. That's where the stress is. If it's a hydrostatic pressure, it's squeezing the volume element so it has to be, by our definition, acting in a negative x1 direction on a surface whose normal is plus x1. So minus P, minus P, minus P, represents a hydrostatic pressure.

And now, unless you've seen it before, or you're reading the notes instead of listening to me, let me give you a really strange-looking one. What is this stress tensor? Where sigma 22 is opposite in sign but equal in magnitude to sigma 11.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah. Good. The way to see that geometrically is the following. What we're doing is, we're pushing with a force per unit area in this direction. That would be sigma 11, and then we squeeze in the orthogonal direction. And that is a sigma 22, which is negative. If we rotate the solid by 90 degrees, then we've got this, plus this, acting on this face, same forces per unit area as we have here. And that is a net force per unit area like this.

On the opposite surface, we've got this force per unit area and this force per unit area, component going in this direction. And that is going to be a force that goes like this. That's going to be a force per unit area like this, and similarly, this combined with this is going to be a force acting in the reverse direction. That would be, again, this with this, and this would be this with that. So clearly, that is a body that's being subjected to pure sheer, and if we rotated the axes x1 and x2 by 45 degrees to an x1 prime and x2 prime like this, and then transformed the tensor, it would take the form 0 sigma sigma 0 0.

And then, what do we want now? This would be the only thing we get. Except I should be careful to put a sigma prime on this because, when we change axes, these sigmas will not be of the same magnitude as the original ones. There's going to be a square root of 2 coming from cosine of 45 degrees in there. But, anyway, just rotating the axes 45 degrees of as I've indicated schematically here with vector sums is going to give me two off diagonal terms equal in magnitude that would correspond to sigma 12 and sigma 21.

So those are some special forms of a stress tensor when you place it in a diagonalized form. But let's conclude this part of our discussion by observing that everything that we've said about second-rank tensors holds for the stress tensor. For example, you can always select new axes that put the tensor in diagonal form. So that's one general property. You can define a stress quadric, and we'll do that with the equation sigma ij xi xj equals 1. Why not? It's a second-ranked tensor. We can take the elements and construct a surface from it.

This quadric will have the properties of any other quadric for a second-rank tensor. In particular, if we construct the quadric, it's now for sure going to be able to be either an ellipsoid, an hyperboloid of one or two sheets, or in this case, even an imaginary ellipsoid for a stress that's everywhere compressive. The quadric could have its diagonal elements all negative.

The radius of the stress quadric in a particular direction is going to be the 1 over the square root of the property in that direction. What does that mean, stress in a given direction? Well, let's again remember how we have defined the stress tensor. A force per unit area has three components, k1, k2, k3, and they're equal to sigma ij times lj.

So the value of the property in this direction is going to be the value of the property in the direction defined by [? elsa ?] j and the values of [? elsa ?] j are components of the unit vector that point in that direction. So if this is the direction of k, [? off ?] like this, the value of this quadric in this direction is going to be the value of the part of k that's parallel to the radius vector over the magnitude of the radius vector. But the [? radius ?] vector is a unit vector, so it's simply going to be equal to k parallel. And if k parallel is the part of the vector that is along the surface normal, that is going to be the component of k which is purely tensile. This is a force per unit area, has parts that are, components that are normal to the surface normal. Those are shear components. The component that's directly along the normal to the surface is going to be a purely tensile component. And, therefore, the radius of the quadric in a given direction gives us the tensile component of the force density vector k that is transmitted across the face-- so let's not say face, but interface-- whose normal is in the direction of R.

So this is the radius vector. That's the direction of the normal to a surface and 1 over the radius of-- The radius of the quadric in that direction is going to be 1 over the square root of the tensile component of stress that's transmitted across an interface in this direction.

How are we doing on time? Any questions at this point?

AUDIENCE: Can you explain how you know that's [INAUDIBLE]?

PROFESSOR: Remember what our radius normal property said. If we have a tensor like good conductivity, where a current density is related to an applied electric field, [? essa ?] j and we've-- unfortunately, that's segment 2. This is the conductivity tensor. What we said is that, if we take the components of the tensor, sigma ij, and construct the surface sigma ij xij equals unity, that will be some sort of quadratic form that has the property that the radius in any given direction is 1 over the square root of the value of the property, sigma conductivity in this case. Bad case. I shouldn't have picked things where the sigma is also involved. And the value of the property in the given direction is going to be the parallel component of what happens over the magnitude of what you do.

So in the case of electrical conductivity, if we apply a field in this direction, the value of the property is going to be that part of the current flow, which is parallel to the electric field, divided by the magnitude of the electric field. So that is a relation that you know by heart and have come to not only know, but to love.

So what are we saying here? For the stress tensor, we can take the elements of the stress tensor and construct a quadric. And the thing that's related here is a force per unit area that we're applying to the surface of the crystal, and that's transmitted through the volume of the crystal by the relation that we have defined as the stress tensor. So, unfortunately, again, sigma Ij. And now what is involved are the direction cosines of the normal to the surface.

So what is the applied vector? The applied vector is a unit vector, and this is the normal to the surface. And the surface in question would be a surface like this. The thing that results is a force per unit area k. So what we're doing is taking k, the value of sigma, in that direction. A scalar quantity will be the part of k that's parallel to the normal vector, divided by the magnitude of the unit vector. And that's, a magnitude of n is 1, so that's just k parallel.

So this is the surface. This is its normal. And what the radius of the quadric is going to give us is that part of the force density, which is the force density vector resolved onto the normal vector. And this is a tensile. This is the part of the tensile stress that's transmitted. This is the total stress that has a sheer part and a tensile part. The radius of the quadric is going to give you the part that's transmitted across the surface in this orientation. That is purely a tensile force.

That make sense? I should have done it without sigmas being present in both tensors.

So everything you want to know about stress that's applied to a solid is contained in a second-rank tensor that is symmetric and obeys all of the characteristics that we derived earlier for property tensors, even though this is a field tensor, and not a property tensor.

Let us then turn to the next thing that I want to discuss, again something you've heard before. But we'll do it within the context of the tensor algebra that we've developed before. And this is the concept of strength. And let me introduce the subject by a one-dimensional analog. Suppose we have a very thin elastic band fastened to some immovable surface, and this will be the direction x. And then we pull on it. And here again, we have to be complete and note that there are several different kinds of behavior. One type of behavior is a case where the elastic band stretches, and if there's some point P on the initial state that point P will move to a point P prime that is displaced from the original location by a vector U.

There's another point Q part way along the elastic band. If the thing that we have fastened to the elastic band slipped, and there's no deformation at all, Q would be displaced by the same amount U if there was just translation of the elastic because it wasn't fastened down solidly. But if it's stretching here between this point and this point, it's going to stretch up here, too. And, in general, it will move over to some point Q prime, and this displacement vector is going to be the original U plus delta U. And what we care about in defining strength is not the absolute coordinates, but the interval between these points. This separation is U, and this separation is U plus delta U.

Now this is a common, but not a necessarily unique, sort of behavior. This is something where the displacement does not change along the length of the elastic bands. This is something that's called homogeneous deformation. And this is characterized by a displacement view that varies linearly with position along the elastic band. Does something like this.

It's possible, in a [? grade ?] of material, if you'd like a real example, that U, as a function of distance along this one-dimensional object, does something like this. Why not? That's what might happen for example in-- that's almost an oxymoron, a one-dimensional elastic band whose cross-sectional area changes with distance in that one dimension. So this is homogeneous deformation, as opposed to inhomogeneous deformation.

There's another type of behavior, and this is apparent if you place a thin piece of metal or plastic, let's say, between the pair of grips in the Instron machine, and after deformation, it does something like this. There's an inhomogeneous deformation, which is to referred to colloquially as necking.

A very good example of that is what happens when you grab hold of some cheap plastic shrink wrap that has been shrunk around a package that has been mailed to you. And you want to get it off, so you grab it. And it doesn't deform elastically. What happens is that it necks in places where there's a stress concentration. And you get something exactly like this.

And if we were to indicate that sort of behavior, U as a function of position would be one U, and then it would increase very rapidly. The [? UDX ?] would be very much larger than the places which had uniform elastic deformation, so it would be a nonlinear behavior of this sort. And that would be necking.

If you've never noticed that sort of behavior before, I give you an assignment tonight of going home and getting a piece of Saran wrap from your kitchen drawer and grabbing hold of it and pulling it. And it deforms non-uniformly in a fashion that's indicative of necking.

OK, it's just about five of. Why don't I stop there. And what I would like to do next is to cast this relation, which is a one-dimensional stream, into a three-dimensional situation. And we'll look at some of the properties of the description of a body that's been deformed in three dimensions.