PROFESSOR: All right, I would like to then get back to a discussion of some of the basic relations that we have been discussing. We didn't get terribly far, but I'd like to start with the Cartesian coordinate system that we set up. Rather than using x, y, and z, I'm labeling the axes x1, x2, and x3. And we'll see that the subscripts play a very useful role in the formalism we're about to develop.

Now, the first thing we might want to specify in this coordinate is the orientation of a vector and its components. So let's suppose that this is some vector P. And what I will do to define its orientation is to use the three angles that the vector makes, or the direction makes with respect to x1, x2, x3. And we could define these angles as theta1, that's the angle between the direction and x1, theta2, the angle between our direction or our vector and x2, and finally, not surprisingly, I'll call this one theta3.

So the three components of the vector could be written as P1, the component along x1 is going to be the magnitude of P times the cosine of theta1. The x2 component of P would be the magnitude of P times the cosine of theta2. And P3, the third component, would be the magnitude of P times the cosine of theta3.

Now, we will have so many relations that involve the cosine of the angle between a direction and one of our reference axes that it is convenient to define a special term for the cosines of these angles. So I'll define this as magnitude of P times the quantity l1, magnitude of P times l2, magnitude of P times l3, which is a lot easier to write. And we will define these things as the direction cosines.

With these equations it's easy to attach some meaning to the direction cosines. Suppose we had a vector of magnitude 1, something that we will refer to as a unit vector. And if we put in magnitude of P equal to 1, it follows that l1m l2m l3 are simply the components of a unit vector in a particular direction along, obviously, x1, x2, and x3, respectively. Trivial piece of algebra, but it attaches a physical and geometric significance to the direction cosines.

Now, the vector is something that could represent a physical quantity. In any case, it is something that is absolute. And it sits embedded majestically, relative to some absolute coordinate system. The magnitudes of the components P1, P2, and P3 will change their values if we would decide to change the coordinate system that we're using as our reference system.

So the next question we might ask is, suppose we change the coordinate system to some new values, x1 prime, x2 prime, and x3 prime? And I'll illustrate my point with just a two dimensional analog of this. This is x1, and this is x2. And this is my vector P.

And I change x1 to some new value, x1 prime, and change x2 to some new orientation, x2 prime. Then clearly the component of P on x1 has changed its numerical value if I refer it to x1 prime instead. And similarly, this value would be the component P2. If I change the direction of x2 and draw a perpendicular to x2, this would be P2 prime. So if I change coordinate system along the fashion I suggested, the three components of a vector, P1, P2, P3, are going to change to some new values, P1 prime, P2 prime, P3 prime.

OK. So the question I'd like to address next is given the change of coordinate system, and given the three components of P in the original coordinate system, how do I compute the values of the new components P1 prime, P2 prime, P3 prime? I'll say it in words, and then we'll define a mechanism for specifying the change in coordinate system.

What I'll say is-- and this was apparent in the sketch that I just erased-- the new component of the vector P1 prime is simply going to be the sum of the components of P1, P2, and P3 along the new x prime direction. So I'm saying that this is going to be the sum of the component of P1 along x1 plus the component of P2 along x1 prime and the component of P3 along x1 prime. So in short, I'm doing nothing more complicated than saying, I can get the values of the new components if I take the vector P, split it up into its three parts, and then find the component of each of these three parts along the x1 prime access, then do the same thing for the x2 prime axis, and then the same thing for x3 prime.

So I'm going to need, now, a notation for a change in a three-dimensional Cartesian coordinate system. So here is x1, here is x2, and here is x3. And I will change them. And again, I'm always keeping the coordinate system Cartesian. So here's an x1 prime, here's an x2 prime, and then x3 prime will point out in some direction like this. I don't want a prime on that.

So I'm going to say now that the component of P along the new x1 prime is going to be the magnitude of P1 times the cosine of the angle between x1 and x1 prime, plus P2 times the cosine of the angle between x1 and-- am I doing this right? P1 onto x1 prime. And I want P2 onto x1 prime. So this is going to be the angle between x2 and x1 prime plus P3 times the cosine of the angle between x3 and x1 prime.

Well, we used C's or l earlier on to represent a direction cosine. Let me define Cij as the cosine of the angle between x1 prime, xi prime, and x sub j. So that means I can write this expression here in this nice compact form. With our definition of direction cosines, I can say that P1 prime is going to be equal to C1 1 times P1 plus C1 2 times P2 plus C1 3 times P3.

Can write that P2 prime in the same way. It's going to be the cosine of the angle between P2 prime and P1 and x1, plus the cosine of the angle between x2 prime and x2 times P2 plus C2 3 which is the cosine of the angle between x2 prime and x3, times the component P3. And in a very similar fashion, P3 prime will be C3 1 P1 plus C3 2 times P2 plus C3 3 times P3. So here is the way a vector will transform.

And we can write this compactly in matrix form. We can say that P sub i prime, where this is a column matrix, one by three, is going to be equal to Cij, a three by three matrix times the original components of the vector P sub j. And just to cement the notation that we're using, if I put my old axes up here, x1, x2, x3, and put the new axes, x1 prime, x2 prime, x3 prime, down this way, then the cosine of the angle between the quantities that are in this column and the quantities that are in this row would be C1 1, C1 2, C1 3, C2 1, C2 2, C2 3, C3 1, C3 2, C3 3. Nothing fancy except the notation. It's the description of some very simple geometry. This array, Cij, is something that I will refer to as a direction cosine scheme.

Let me pause here and see if that's all sunk in, whether you have any questions on this. One of the nasty properties of what we're going to be doing for the next month or so is that the notions are really very, very simple, but the notation is horribly cumbersome and complex. So it takes a bit of getting used to in application to actual real cases before you feel fully at home with it. OK. Just a matter of definition so far.

Let me note that this direction cosine array is going to be useful for defining how a vector changes as we go from the original coordinate system to a new coordinate system. But this direction cosine array will also tell us how the axes in one coordinate system are related to the axes in the new coordinate system. It follows from the fact that the axes themselves can be regarded as unit vectors. And we said that the components of a unit vector are the direction cosines of that vector, relative to a coordinate system.

So let's ask, what are the new components of x1 in terms of the original axes unprimed. Well, x1 prime is going to be the unit vector x1 times the cosine of the angle between x1 and x1 prime, plus the unit vector along x2 prime times the cosine of the angle between x1 prime, and x2. And that's C1 2. Plus x3 regarded as a unit vector times the cosine of the angle between x1 prime and x3. So we can actually write an equation for unit vectors along each of our new axes. And they will go as C1 1 times x1, C1 2 times x2, C1 3 times x3, plus C2 2 times x2 plus C2 3 times x3 times x3 prime. And x3 primal will be C3 1 x1 plus C3 2 x2 plus C3 3 x3. So this, then, is an equation between the unit vectors along the three reference axes in the new coordinate system relative to those in the original coordinate system. And the direction cosine scheme does the job.

OK. We could, using the same argument, give the array that specifies the reverse transformation. If we would change our mind, for example, and say we don't like what we've done, let's write the original coordinate system x1, x2, and x3 in terms of the unit vectors along the new axes. And we can use exactly the same array.

We can say that the original x1, in terms of the three new axes, x1 prime, x2 prime, and x3 prime, is going to involve the cosine of the angle between x1 prime and x1, and that is C1 1, plus the cosine of the angle between x2 prime and x1, and that's C2 1, plus the cosine of the angle between x3 prime and x1, and that C3 1. If we continue on this, if you have the idea, the angle x2 is going to be given in terms of x1 prime, x2 prime, and x3 prime, as the cosine of the angle between x2 and x1 prime, and that is C1 2. Here we want the cosine of the angle between x2 and x2 prime, and here the cosine of the angle between x3 prime and x2.

And you can see the way this is playing out. C1 3 times x1 plus C2 3, x2 prime plus C3 3 times x3 prime. So there's the reverse transformation, using the same array of coefficients as we did the first time. So it turns out if we write this symbolically in a compact form, xi prime is given by Cijx sub j. And the reverse transformation using the same direction cosine says that xi is going to be Cji times x sub j prime. In other words, the reverse transformation, let's write it as Cij minus 1, the inverse transformation, turns out to be simply Cji.

And that, in matrix algebra, is written as the transpose of the original array of coefficients. And transpose is either given by a squiggle, a tilde on top of the matrix. Some people like to use a superscript T. But we'll use this particular notation. But you can see either notation used to indicate the transpose.

The array Cij, which has this property, and it also has another property which I won't bother to prove, but the determinant of Cij is equal to 1. And this is something called a unitary matrix. Unitary matrix has the property that the inverse matrix is the transpose. We will very, very shortly start writing down numbers for some specific transformations. And then I think that will give us a little facility in doing these manipulations.

Comments or questions? Is this old stuff or old stuff for which the notation is still confusing? All right. Let me point out something that is perhaps apparent to you. And that is that not all nine of these numbers are independent. There are relations between them. And let's point out some of these relations. C1 1, C1 2, C1 2 represent the components of a unit vector along x1 prime, in the original coordinate system of the elements in any row is equal to 1. Because these are the components of a unit vector. And the magnitude of a unit vector is 1.

In the same way, if we look at any column of terms in this matrix, for example, C1 1, C2 1, C3 1, these are terms that represent the cosine of angles between x1 in the original coordinate system and x1 prime, x2 prime, x3 prime, our new coordinate system. So this gives us the magnitude of x1, but x1 is a unit vector. So it follows that the sum of the column C1 1 squared plus C2 1 squared plus C3 1 squared also has to be unity, because that gives us the magnitude of a unit vector along x1, So the sum of the squares of elements in any column of the direction cosine is unity.

These expressions are useful. But they have one ambiguity. That is the cosine of an angle can be either positive or negative, depending on whether the angle is less than 90 degrees or greater than 90 degrees. These relations involve the squares of direction cosines, and therefore we can't tell whether the direction cosine itself is positive or negative. So let me put down a limitation here. And that is we cannot tell the sign.

Every time I point this out to people I wince inside. Because I once spent two weeks trying to debug a computer program, and it wasn't working. And it turns out the reason it wasn't working properly was that I didn't realize that you cannot tell the sign when all you know is the squares of the direction cosines. So I remember this as a rather pointed observation.

Happily, there are other relations among this array of coefficients. This row of terms represents the components of x1 prime in the original coordinate system x1, x2, x3. This row immediately below it represents the components of a unit vector x2 prime relative to the original coordinate system x1, x2, x3. Our coordinate systems are Cartesian. Therefore, this unit vector has to be perpendicular to the unit vector along x2 prime. And that means their dot product has to be 0.

So let me indicate that this way. The unit vector along x1 prime dotted with the unit vector along x2 prime has to be 0. And that dot product is going to be C1 1 times C2 1, that's the product of these two terms, plus C1 2 times C2 2 plus C1 3 times C2 3. And that has to be 0. And this involves only the first product of the direction cosine.

So to make it come out 0 when we add up the magnitudes, we will get the sign of the direction cosine. So this is a much more powerful relation.

And similarly, the product of the coefficients in the first and the third row have to add up to 0. And the second and third row have to be 0. So there are three different relations we can write between products of corresponding coefficients in the rows. So to sum up in words, the sum of the corresponding elements-- of the product of-- in any pair of rows of Cij must be 0.

But we're not done yet. If we look at the columns in this array, this represents the components of x1 relative to x1 prime, x2 prime, x3 prime. And these terms here represent the components of x2 relative to x1 prime, x2 prime, and x3 prime. And for similar reasons, the dot product of those two vectors has to be 0. So we can say that in addition, the sum of any sum of pairs of corresponding coefficients in any pair of columns must be 0.

So we're working here on the direct matrix of the transformation. We've seen that the reverse relation, the inverse matrix of Cij is Cij transpose. And therefore the inverse matrix has to have this same relationship that the products of terms and rows or columns, any pair of rows or columns has to be 0.

Now, there's one other pair of relations among the coefficients, which is not quite so geometrically obvious. And I won't attempt to prove it. I'll just state it. I said a moment ago that these are unitary matrices. The determinant of the coefficient Cij then has to be unity. But interestingly, it will be plus 1 if one goes from a right-handed system to a right-handed system. That is to say the set of axes x1, x2, x3 might be right-handed. And if the new coordinate system x1 prime, x2 prime, x3 prime is also right-handed, then the determinant of coefficients is plus 1.

On the other hand, if one goes from a right-handed system to a left-handed reference system or from a left-handed one to a right-handed coefficient, then, interestingly the determinant of coefficients is minus 1. So the determinant of the matrix of the transformation is plus 1 if you go to coordinate system of the same chirality. It's equal to minus 1 if you go to a coordinate system of changed chirality.

All right, so to repeat something I said at the outset but which you now probably truly believe, the elements in the direction cosine scheme that get you from one coordinate system to another have lots of inter-relations. And all of these coefficients are not independent. There are these relations that couple them. How are we doing on time?

We mentioned last time that a large collection of physical properties of materials are properties that relate a pair of vectors. So let me, to make this specific, talk about a particular physical property, electrical conductivity. And electrical conductivity relates a current density vector, and that it charge per unit area per unit time to an applied vector, and that vector is the electric field vector. And the electric field has units of volts per unit length, so volts per meter in MKS.

And provided the electric field that's supplied is not too strong, it turns out that every component of the current flow is given by a linear combination of every component of the applied electric field. So the flow of current along x1 will be given by a proportionality constant, an element sigma 1 1 times the x1 component of the electric field. Let me write it out the first couple of times we do this. Sigma 1 1 times E1 plus sigma 1 2 times E2 plus sigma 1 3 times E3.

J2 will be sigma 2 1 times E1 plus sigma 2 2 times E2 plus sigma 2 3 times E3. And J3 will be equal to sigma 3 1 times E1 plus sigma 3 2 times E2 plus sigma 3 3 times E3. Looks formally like the relation between unit vectors that define a coordinate system. Number of subscripts is the same, but actually this is something that's completely different. It's dealing with vectors that have some physical significance. So in compact reduced subscript notation, this is the definition of electrical conductivity. This matrix that relates the electric field vector to the current density vector is said to be a tensor of the second rank.

OK, tensor. First thing you might say, why do you call it a tensor, dummy? It's a matrix. It's a plain old matrix. There's a subtle but very important difference. A tensor is a matrix with an attitude. And I'll make the distinction clear a little bit later on. But there are tensors also of higher rank. These expressions where summation over repeated subscripts is implied can hide, as I indicated last time, some absolutely horrendous polynomials.

But tensor at very least is a term that makes the faces of all who hear it pale, and makes the knees of even the very strong to weaken. And in case you don't believe that, I'll show you what I have to wear whenever I give these lectures. And consequently it's kind of scuzzy and worn out. But I have to put on these knee braces from wobbling braces. And you can see what it says on here. "Tensor." So that's a consequence of this frightening definition that we've just made.

Let me next set the stage for what we ought to do next. E sub j represents the components of an electric field, x1, x2, x3, in a first coordinate system. ji represent the components of the current flow in a coordinate system, x1, x2, x3. If we were to change coordinate system for any reason, these three numbers would wink on and off. Some might go negative. The magnitudes would change. And as a result, the components of the current flow would have to do the same thing. Because the components of these vectors, without changing anything physically, have to change their numerical values if we refer them to a new set of reference axes.

If we change coordinate system and these numbers change, and if we change coordinate system, these numbers change, we're still applying field in the same direction. The current still flows in the same direction. But the components we use to define these two vectors change. And it follows just algebraically, the elements of the tensor have to change and link into different values. It follows automatically. So a question, then, is that if we have a coordinate system, x1, x2, x3, and we change it into a new coordinate system, x1 prime, x2 prime, x3 prime, then j sub i changes to some new values, j sub i prime. E sub j changes to some new values, E sub j prime. And therefore, of necessity, sigma i j, the conductivity tensor, has to change to new values sigma ij prime.

So I'll let you rest up to brace yourself for this. The question is, how can we get sigma ij prime, the nine elements of the tensor in the new coordinate system, in terms of the direction cosine scheme that defines this transformation and in terms of the elements of the original conductivity tensor? And this, my friends, is what makes a tensor a tensor and not a matrix. I can write a matrix for you, a really lovely matrix. Let's put in some elements here. Let's put in 6.2, square root of minus 1e, and 23. And as other elements, I'll put in pi 23.4, 6, and 0. It's a perfectly good matrix. It's just an array of numbers, any numbers, real or imaginary, or whatever I like. So this is a matrix.

What a tensor is, is a matrix for which a law of transformation is defined. And that's what makes a tensor a tensor. What does it mean to take this two-by-four matrix that I just wrote down? How do I transform that to a different coordinate system? It's meaningless, just an array of numbers. It's an array of numbers that has some useful properties, like matrix multiplication and the like.

But to talk about transformation of this set of four ridiculous numbers to a new coordinate system is something that's absolutely meaningless. Not so for something like conductivity or the piezoelectric moduli or the elastic constants. These change their values. There's a law of transformation when we go from one Cartesian reference system to another.

So what we will do when and if you return is to derive a law for transformation for second-rank tensors, and then, by implication, look at higher-rank tensors and decide how they would transform. But why would you want to do this? Why would you want to muck things up and have to worry about transforming these numbers?

Well, let me give you just one simple example. Suppose we had conductivity of a plate, of a crystal. And what would you do? You'd measure it relative to a set of axes, which, if you have a little fragment of crystal, you have no reference system. So say that the axes x of i are taken relative to the lattice constants of the material, so relative to the edges of the unit cell, possibly.

Then you decide that this material really has some useful properties, and you would like to cut a piece out of it so that you get a plate for which the maximum conductivity in that plate is in a direction normal to the plate. So you know just what sort of plate you want to cut out. You know what the direction cosines are. But once you've cut a plate from the crystal, the tensor relative to the old axes, x1, x2, x3, is not going to be terribly useful. You're going to want to find the tensor relative to this as one set of axes, and these perhaps as a new set of axes within the plane of the plate.

So there's a good example. Cut a piece from a crystal and cut that piece so that the extreme values are along x, y, and z for the new coordinate system. Then you will be faced with the necessity of transforming the tensor from one coordinate system to another one. Or you might measure the thermal conductivity tensor. You might want to cut a rod out of the material so that the maximum conductivity or the minimum thermal conductivity is along the direction of the rod. You might want to use that as a push rod to hold a sample in position and not have it be a big heat sink for the temperature that's inside of your sample chamber.

So I've hopefully convinced you that there are lots of cases where it would be necessary and convenient to transform the tensor that describes a property to a new coordinate system. All right, so let us take our break now. Some internal clock always tells me when it's five of the hour, unless I get really excited about something. And it is indeed that time now. So let's stop.