18.01 Calculus
Exam 5 at 2:00pm sharp
Tuesday, November 29, 2005

Jason Starr
Fall 2005

Inverse Substitutions

Let a be a positive real number. For antidifferentiating a rational expression $F(x, y(x))/G(x, y(x))$ where $y(x)$ is one of $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$ or $\sqrt{x^2 + a^2}$, use either an inverse trigonometric, inverse hyperbolic or inverse rational substitution. Altogether, there are 9 possibilities.

$y(x)$	$\sqrt{a^2 - x^2}$	$\sqrt{x^2 - a^2}$	$\sqrt{x^2 + a^2}$
Trig. subst.	$x = a\sin(\theta)$	$x = a\sec(\theta)$	$x = a\tan(\theta)$
Range of θ	$-\pi/2 \leq \theta \leq \pi/2$	$\begin{cases} 0 \leq \theta < \pi/2, & x > 0, \\ \pi/2 < \theta \leq \pi, & x < 0 \end{cases}$	$-\pi/2 < \theta < \pi/2$
$y(\theta)$	$a\cos(\theta)$	$\begin{cases} +a\tan(\theta), & x > 0, \\ -a\tan(\theta), & x < 0 \end{cases}$	$a\sec(\theta)$
dx	$a\cos(\theta)d\theta$	$a\sec(\theta)\tan(\theta)d\theta$	$a\sec^2(\theta)d\theta$
θ	$\sin^{-1}(x/a)$	$\cos^{-1}(a/x)$	$\tan^{-1}(x/a)$
Hyper. subst.	$x = a\tanh(t)$	$x = \begin{cases} +a\cosh(t), & x > 0, \\ -a\cosh(t), & x < 0 \end{cases}$	$x = a\sinh(t)$
Range of t	$-\infty < t < \infty$	$t \geq 0$	$-\infty < t < \infty$
$y(t)$	$a\mathrm{sech}(t)$	$a\sinh(t)$	$a\cosh(t)$
dx	$a\mathrm{sech}^2(t)$	$\begin{cases} +a\sinh(t)dt, & x > 0, \\ -a\sinh(t)dt, & x < 0 \end{cases}$	$a\cosh(t)dt$
t	$\frac{1}{2}\ln((a+x)/(a-x))$	$\ln((\lvert x\rvert + \sqrt{x^2 - a^2})/a)$	$\ln((x + \sqrt{x^2 + a^2})/a)$
Rat'l. Subst.	$x = a(z^2 - 1)/(1 + z^2)$	$x = \begin{cases} +a(1+z^2)/(2z), & x > 0, \\ -a(1+z^2)/(2z), & x < 0 \end{cases}$	$x = a(z^2 - 1)/(2z)$
Range of z	$z \geq 0$	$z \geq 1$	$z > 0$
$y(z)$	$a(2z)/(1 + z^2)$	$a(z^2 - 1)/(2z)$	$a(1 + z^2)/(2z)$
dx	$a(4z)/(1 + z^2)^2 dz$	$\begin{cases} +a(z^2-1)/(2z^2)dz, & x > 0, \\ -a(z^2-1)/(2z^2)dz, & x < 0 \end{cases}$	$a(1 + z^2)/(2z^2)dz$
z	$\sqrt{(a+x)/(a-x)}$	$(\lvert x\rvert + \sqrt{x^2 - a^2})/a$	$(x + \sqrt{x^2 + a^2})/a$

To go between the 3 substitutions, use,

$$z = e^t \text{ and } z = \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right).$$

It is worth noting, another common choice for the rational substitutions is to use,

$$w = \frac{z-1}{z+1}, \quad z = \frac{1+w}{1-w}.$$

Then to go between the 3 substitutions, use,

$$w = -\tanh\left(\frac{t}{2}\right) \text{ and } w = \tan\left(\frac{\theta}{2}\right).$$

This gives the grid,

$y(x)$	$\sqrt{a^2 - x^2}$	$\sqrt{x^2 - a^2}$	$\sqrt{x^2 + a^2}$
Rat'l. Subst.	$x = a(2w)/(1+w^2)$	$x = \begin{cases} +a(1+w^2)/(1-w^2), & x > 0, \\ -a(1+w^2)/(1-w^2), & x < 0 \end{cases}$	$x = a(2w)/(1-w^2)$
Range of w	$-1 \leq w \leq 1$	$0 \leq w < 1$	$-1 < w < 1$
$y(w)$	$a(1-w^2)/(1+w^2)$	$a(2w)/(1-w^2)$	$a(1+w^2)/(1-w^2)$
dx	$2a(1-w^2)/(1+w^2)^2 dw$	$\begin{cases} -a(4w)/(1-w^2)^2 dw, & x > 0, \\ +a(4w)/(1-w^2)^2 dw, & x < 0 \end{cases}$	$2a(1+w^2)/(1-w^2) dz$
w	$x/(a+\sqrt{a^2-x^2})$	$\pm\sqrt{(\lvert x\rvert - a)/(\lvert x\rvert + a)}$	$x/(a+\sqrt{a^2+x^2})$