18.014 Problem Set 10 Solutions

Total: 12 points

Problem 1: Evaluate

a. $\lim_{x\to\infty} \dfrac{\log(a+be^x)}{\sqrt{a+bx^2}}$ b. $\lim_{x\to 1^-} \log(x)\log(1-x)$.

$\boxed{\text{Solution}}$ (4 points) For part (a), we first evaluate the limits $\lim_{x\to\infty} \frac{\log(a+be^x)}{x}$ and $\lim_{x\to\infty} \frac{x}{\sqrt{a+bx^2}}$. The second limit can be computed as follows:

$$\lim_{x\to\infty} \frac{x}{\sqrt{a+bx^2}} = \lim_{x\to\infty} \sqrt{\frac{x^2}{a+bx^2}} = \lim_{t\to 0^+} \sqrt{\frac{1}{at^2+b}} = \frac{1}{\sqrt{b}}.$$

For the first limit, we use L'Hopital's rule to get

$$\lim_{x\to\infty} \frac{\log(a+be^x)}{x} = \lim_{x\to\infty} \frac{be^x/(a+be^x)}{1} = \lim_{x\to\infty} \frac{b}{ae^{-x}+b} = \lim_{t\to 0^+} \frac{b}{ae^t+b} = 1.$$

Now, we know

$$\lim_{x\to\infty} f(x)g(x) = \lim_{x\to\infty} f(x) \lim_{x\to\infty} g(x)$$

whenever the two limits on the right hand side exist. Putting it all together, we have

$$\lim_{x\to\infty} \frac{\log(a+be^x)}{\sqrt{a+bx^2}} = \frac{1}{\sqrt{b}}.$$

For part (b), note $\lim_{x\to 1^-} \frac{1}{\log(x)} = \infty$ and $\lim_{x\to 1^-} \log(1-x) = \infty$. Using L'Hopital's rule, we see

$$\lim_{x\to 1^-} \frac{\log(1-x)}{1/\log(x)} = \lim_{x\to 1^-} \frac{-1/(1-x)}{-1/\log(x)^2 x} = \lim_{x\to 1^-} \frac{x\log(x)^2}{(1-x)}.$$

Applying L'Hopital's rule again, we get

$$\lim_{x\to 1^-} \frac{2x\log(x)(1/x)+\log(x)^2}{-1} = \lim_{x\to 1^-} (-2\log(x) - \log(x)^2) = 0.$$

Problem 2: For $|x|<1$, show

a. $\displaystyle\sum_{n=1}^{\infty} n^2 x^n = \frac{x^2+x}{(1-x)^3}$ b. $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n = \frac{1}{(1-x)^3}$.

$\boxed{\text{Solution}}$ (4 points) First, we show (b). Then we show (a). Recall

$$\frac{1}{(1-x)} = \sum_{n=0}^{\infty} x^n$$

for $|x| < 1$. Taking the derivative of both sides, we get

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$

for $|x| < 1$. Taking another derivative yields

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$$

for $|x| < 1$. Shifting the index and dividing by two, we get

$$\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^n$$

for $|x| < 1$. This is the statement for part (b). For part (a), we multiply both sides by $x + x^2$ to get

$$\frac{x+x^2}{(1-x)^3} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^{n+1} + \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^{n+2}$$

for $|x| < 1$. Shifting the indices and summing the terms, we get

$$\frac{x+x^2}{(1-x)^3} = \sum_{n=1}^{\infty} \frac{(n+1)(n)}{2} x^n + \sum_{n=2}^{\infty} \frac{(n)(n-1)}{2} x^n = 1 + \sum_{n=2}^{\infty} n^2 x^n = \sum_{n=0}^{\infty} n^2 x^n$$

when $|x| < 1$. This is the answer for part (a).

Problem 3: (a) Given that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x, show

$$\sum_{n=0}^{\infty} \frac{n^2 x^n}{n!} = (x^2 + x)e^x$$

for all x.
(b) Compute $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!}$.

$\boxed{\text{Solution}}$ (4 points) (a) We start with $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and we multiply both sides by $x + x^2$ to get

$$(x+x^2)e^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} + \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} = x + \sum_{n=2}^{\infty} \left(\frac{x^n}{(n-1)!} + \frac{x^n}{(n-2)!} \right)$$
$$= x + \sum_{n=2}^{\infty} \left(\frac{nx^n}{n!} + \frac{n(n-1)x^n}{n!} \right) = x + \sum_{n=2}^{\infty} \frac{n^2x^n}{n!} = \sum_{n=1}^{\infty} \frac{n^2x^n}{n!}.$$

(b) Differentiating both sides of the equation in part (a) yields

$$(x^2+3x+1)e^x = \sum_{n=1}^{\infty} \frac{n^3x^{n-1}}{n!}.$$

Multiplying both sides by x gives

$$(x^3+3x^2+x)e^x = \sum_{n=1}^{\infty} \frac{n^3x^n}{n!}.$$

Plugging in $x = 1$, we have

$$5e = \sum_{n=1}^{\infty} \frac{n^3}{n!}.$$

Bonus: A function f is called uniformely continuous if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all x, y with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. Prove that every continuous function on a closed interval $[a, b]$ is uniformely continuous.

$\boxed{\text{Solution}}$ (4 points) Given $\epsilon > 0$, by the small span theorem (Notes H.1) there exists a partition $x_0 < x_1 < \cdots < x_n$ such that $|f(x) - f(y)| < \frac{\epsilon}{2}$ whenever $x_i \le x, y \le x_{i+1}$ for some i. Put

$$\delta = \min_{i=0}^{n-1} \{|x_{i+1} - x_i|\}$$

and suppose $|x - y| < \delta$. Without loss of generality, assume $x < y$ and assume $x_{i-1} \le x < x_i$. Since $y - x < \delta$ and $x_{i+1} - x_i \ge \delta$, we must have $x_{i-1} \le y \le x_{i+1}$. There are two cases. If $x_{i-1} \le y \le x_i$, then

$$|f(y) - f(x)| < \frac{\epsilon}{2} < \epsilon$$

by our hypotheses and the small span theorem. If $x_i < y \le x_{i+1}$, then we use the small span theorem twice to get

$$|f(y) - f(x)| \le |f(y) - f(x_i)| + |f(x_i) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Regardless of case, we proved the desired result.