18.014 Problem Set 9 Solutions

Total: 24 points

Problem 1: Integrate
(a)
$$\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)}.$$
(b)
$$\int \frac{dx}{x^4-2x^2}.$$

$\boxed{\text{Solution}}$ (4 points) (a) We use the method of partial fractions to write
$$\frac{1}{(x-2)^2(x^2-4x+5)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{(Cx+D)}{x^2-4x+5}.$$
Making a common denominator yields the expression
$$A(x-2)(x^2-4x+5) + B(x^2-4x+5) + (Cx+D)(x-2)^2 = 1.$$
Plugging in $x=2$, we get $B=1$. Taking the derivative and plugging in $x=2$ yields $A=0$. Solving for C and D, we have $(x^2-4x+5)+(Cx+D)(x-2)^2 = 1$. Comparing x^3 terms, we see $C=0$. Comparing x^2 terms, we see $D=-1$. Thus, we have
$$\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)} = \int \frac{dx}{(x-2)^2} - \int \frac{dx}{x^2-4x+5}.$$
(Give yourself a pat on the back if you noticed from the start that the terms x^2-4x+5 and x^2-4x-4 differed by 1 and got this decomposition without solving for A,B,C, and D).
Now the first integral is $-(x-2)^{-1}+C$. To do the second integral, we complete the square and substitute $u=x-2$
$$\int \frac{dx}{x^2-4x+5} = \int \frac{dx}{(x-2)^2+1} = \int \frac{dx}{u^2+1} = \arctan u + C = \arctan(x-2) + C.$$
Thus, our final answer is
$$\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)} = -(x-2)^{-1} - \arctan(x-2) + C.$$

(b) Hopefully, after doing part (a), everyone observed

$$\frac{1}{x^2(x^2-2)} = \frac{1}{2}\left(\frac{1}{x^2-2} - \frac{1}{x^2}\right) \text{ and } \frac{1}{x^2-2} = \frac{1}{2\sqrt{2}}\left(\frac{1}{x-\sqrt{2}} - \frac{1}{x+\sqrt{2}}\right).$$

If not, one can figure this out using partial fractions. Now, we have

$$\begin{aligned}\int \frac{dx}{x^4-2x^2} &= \int \left(\frac{1}{4\sqrt{2}}\left(\frac{1}{x-\sqrt{2}} - \frac{1}{x+\sqrt{2}}\right) - \frac{1}{2x^2}\right) dx \\ &= \left(\frac{1}{4\sqrt{2}}\left(\log|x-\sqrt{2}| - \log|x+\sqrt{2}|\right) + \frac{1}{2x}\right) + C.\end{aligned}$$

Problem 2: Let $A = \int_0^1 \frac{e^t}{t+1} dt$. Express the values of the following integrals in terms of A:
(a) $\displaystyle\int_{a-1}^{a} \frac{e^{-t}}{t-a-1} dt$ (b) $\displaystyle\int_0^1 \frac{te^{t^2}}{t^2+1} dt$ (c) $\displaystyle\int_0^1 \frac{e^t}{(t+1)^2} dt$ (d) $\displaystyle\int_0^1 e^t \log(1+t) dt$.

$\boxed{\text{Solution}}$ (4 points) For part (a), we substitute $t = -u + a$ to get

$$-\int_1^0 \frac{e^{u-a}}{-u-1} du = -e^{-a} \int_0^1 \frac{e^u}{u+1} du = -e^{-a} A.$$

For part (b), we substitute $u = t^2$ to get

$$\frac{1}{2}\int_0^1 \frac{e^u}{u+1} du = \frac{1}{2} A.$$

For part (c), we integrate by parts to get

$$-\frac{e^t}{(t+1)}\bigg|_0^1 + \int_0^1 \frac{e^t}{t+1} dt = -\frac{e}{2} + 1 + A.$$

For part (d), we integrate by parts to get

$$e^t \log(1+t)\bigg|_0^1 - \int_0^1 \frac{e^t}{1+t} dt = e\log(2) - A.$$

Problem 3: Let $F(x) = \int_0^x f(t) dt$. Determine a formula (or formulas) for computing $F(x)$ for all real x if f is defined as follows:

(a) $f(t) = (t + |t|)^2$.
(b) $f(t) = \left\{ \begin{array}{lll} 1 - t^2 & \text{if} & |t| \leq 1 \\ 1 - |t| & \text{if} & |t| > 1 \end{array} \right\}$
(c) $f(t) = e^{-|t|}$.
(d) $f(t) = \max\{1, t^2\}$.

$\boxed{\text{Solution}}$ (4 points) (a) If $t \leq 0$, then $f(t) = 0$. If $t \geq 0$, then $f(t) = 4t^2$. Hence, if $x \leq 0$ then $F(x) = 0$, and if $x \geq 0$ then

$$F(x) = 4 \int_0^x t^2 dt = \frac{4}{3}x^3.$$

(b) If $x \leq -1$, then

$$F(x) = \int_0^x f(t)dt = \int_0^{-1} (1 - t^2)dt + \int_{-1}^x (1 + t)dt = \left(t - \frac{t^3}{3}\right)\Big|_0^{-1} + \left(t + \frac{t^2}{2}\right)\Big|_{-1}^x$$
$$= -1 + \frac{1}{3} + x + \frac{x^2}{2} + 1 - \frac{1}{2} = -\frac{1}{6} + x + \frac{x^2}{2}.$$

If $-1 \leq x \leq 1$, then

$$F(x) = \int_0^x f(t)dt = \int_0^x (1 - t^2)dt = \left(t - \frac{t^3}{3}\right)\Big|_0^x = x - \frac{x^3}{3}.$$

If $1 \leq x$, then

$$F(x) = \int_0^x f(t)dt = \int_0^1 (1 - t^2)dt + \int_1^x (1 - t)dt = \left(t - \frac{t^3}{3}\right)\Big|_0^1 + \left(t - \frac{t^2}{2}\right)\Big|_1^x$$
$$= 1 - \frac{1}{3} + x - \frac{x^2}{2} - 1 + \frac{1}{2} = \frac{1}{6} + x - \frac{x^2}{2}.$$

(c) If $x \geq 0$, then

$$F(x) = \int_0^x f(t)dt = \int_0^x e^{-t}dt = -e^{-t}\Big|_0^x = -e^{-x} + 1.$$

If $x \leq 0$, then

$$F(x) = \int_0^x f(t)dt = \int_0^x e^t dt = e^t\Big|_0^x = e^x - 1.$$

(d) Note $t^2 \leq 1$ if $|t| \leq 1$ and $t^2 \geq 1$ if $|t| \geq 1$. Hence, $f(t) = 1$ if $|t| \leq 1$ and $f(t) = t^2$ if $|t| \geq 1$. Thus, for $x \leq -1$, we have

$$F(x) = \int_0^x f(t)dt = \int_0^{-1} dt + \int_{-1}^x t^2 = -1 + \frac{x^3}{3} + \frac{1}{3} = \frac{x^3}{3} - \frac{2}{3}.$$

For $-1 \leq x \leq 1$, we have

$$F(x) = \int_0^x f(t)dt = \int_0^x dt = x.$$

And for $x \geq 1$, we have

$$F(x) = \int_0^x f(t)dt = \int_0^1 dt + \int_1^x t^2 dt = 1 + \frac{x^3}{3} - \frac{1}{3} = \frac{2}{3} + \frac{x^3}{3}.$$

Problem 4: Use Taylor's formula to show
(a)

$$\sin(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}x^{2k-1}}{(2k-1)!} + E_{2n}(x), \text{ where } |E_{2n}(x)| \leq \frac{|x|^{2n-1}}{(2n+1)!}.$$

(b)

$$\cos(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} + E_{2n+1}(x), \text{ where } |E_{2n+1}(x)| \leq \frac{|x|^{2n+2}}{(2n+2)!}.$$

$\boxed{\text{Solution}}$ (4 points) (a) By Taylor's formula (Theorem 7.6), we know

$$\sin(x) = \sum_{l=0}^{2n} \frac{(\sin^{(l)}(0))x^l}{l!} + E_{2n}(x).$$

Note $\sin^{2l}(0) = \pm\sin(0) = 0$, $\sin^{4l+1}(0) = \cos(0) = 1$, and $\sin^{4l+3}(0) = -\cos(0) = -1$. Thus, our sum becomes $\sum_{k=1}^{n} \frac{(-1)^{k-1}x^{2k-1}}{(2k-1)!}$. To approximate the error term, we use theorem 7.7 to conclude

$$|E_{2n}(x)| \leq M\frac{|x|^{2n+1}}{(2n+1)!} \text{ where } M = \sup_{|c|\leq|x|} |\sin^{(2n+1)}(c)|.$$

Since $\sin^{(2n+1)}(c) = \pm\cos(c)$, we observe $|M| \leq 1$, and the bound for our error term becomes $|E_{2n}(x)| \leq \frac{|x|^{2n+1}}{(2n+1)!}$ as desired.

(b) This time we observe $\cos^{(2l+1)}(0) = \pm\sin(0) = 0$, $\cos^{(4l)}(0) = \cos(0) = 1$, and $\cos^{(4l+2)}(0) = -\cos(0) = -1$. Plugging this into Taylor's formula yields

$$\cos(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} + E_{2n+1}(x).$$

To approximate the error term, we again use theorem 7.7 together with the bound $|\cos^{(l)}(c)| \leq 1$ for all l. This yields the estimate $|E_{2n+1}(x)| \leq \frac{|x|^{2n+2}}{(2n+2)!}$.

Problem 5: Evaluate the following limits:
(a) $\lim_{x\to 0} \dfrac{x(e^x+1)-2(e^x-1)}{x^3}$ (b) $\lim_{x\to 0} \dfrac{\log(1+x)-x}{1-\cos(x)}$.

$\boxed{\text{Solution}}$ (4 points) (a) We apply L'Hopital's rule (Theorem 7.9). Observe that both the numerator and the denominator are differentiable and take the value zero at 0. We observe $\frac{d}{dx}(x(e^x+1)-2(e^x-1)) = xe^x - e^x + 1$ and $\frac{d}{dx}x^3 = 3x^2$. Each of these functions is differentiable and takes the value zero at 0. Differentiating again, we get $\frac{d}{dx}(xe^x - e^x + 1) = xe^x$ and $\frac{d}{dx}(3x^2) = 6x$. Both of these functions are differentiable and take the value zero at 0. Finally, we differentiate both functions one more time to get $\frac{d}{dx}(xe^x) = e^x + xe^x$ and $\frac{d}{dx}(6x) = 6$. Applying L'Hoptal's rule three times, we get

$$\lim_{x\to 0} \frac{x(e^x+1)-2(e^x-1)}{x^3} = \lim_{x\to 0} \frac{e^x + xe^x}{6} = \frac{1}{6}.$$

The last equality comes from observing that the numerator and denominator are continuous functions and plugging in $x = 0$.

(b) Again, we use L'Hopital's rule (Theorem 7.9). Observe that both the numerator and denominator are differentiable at 0 and take the value zero at 0. Computing, we get $\frac{d}{dx}(\log(1+x)-x) = \frac{1}{(1+x)} - 1$ and $\frac{d}{dx}(1-\cos(x)) = \sin(x)$. Both of these functions are differentiable at 0 and take the value zero at 0. Computing again, we get $\frac{d}{dx}(\frac{1}{1+x} - 1) = -\frac{1}{(1+x)^2}$ and $\frac{d}{dx}\sin(x) = \cos(x)$. Applying L'Hopital's rule twice, we get

$$\lim_{x\to 0} \frac{\log(1+x)-x}{1-\cos(x)} = \lim_{x\to 0} \frac{-1/(1+x)^2}{\cos(x)} = -1.$$

The last equality comes from observing that the numerator and denominator are both continuous at zero and plugging in $x = 0$.

Problem 6: (a) Compute the limit

$$\lim_{x\to 0} \frac{\sin(x) - xe^{x^2} + 7x^3/6}{\sin^2(x)\sin(x^3)}$$

(b) Show that if $|x| < 1$, then

$$\left|e^x - \left(1 + x + x^2/2\right)\right| \leq \left|x^3/2\right|.$$

(c) Show that if $|t| < 1$, then the approximation

$$\int_0^t e^{x^2} dx \sim t + \frac{t^3}{3} + \frac{t^5}{5}$$

involves an error in absolute value no more than $|t^7/14|$.

$\boxed{\text{Solution}}$ (4 points) For part (a), we apply L'Hopital's rule four times. Let $f(x) = \sin(x) - xe^{x^2} + 7x^3/6$ be the numerator and let $g(x) = \sin^2(x)\sin(x^3)$ be the denominator. Note $f'(x) = \cos(x) - e^{x^2} - 2x^2e^{x^2} + 7x^2/2$, $f''(x) = -\sin(x) - 6xe^{x^2} - 4x^3e^{x^2} + 7x$, $f'''(x) = -\cos(x) - 6e^{x^2} - 24x^2e^{x^2} - 8x^4e^{x^2} + 7$, $f^{(4)}(x) = \sin(x) - 60xe^{x^2} + x^3(f_1(x))$, and $f^{(5)}(x) = \cos(x) - 60e^{x^2} + x(f_2(x))$. One sees that f is 5 times differentiable at 0, $f^{(k)}(0) = 0$ for $k = 0, 1, \ldots, 4$, and $f^{(5)}(0) = -59$. Next, we write $g(x) = h_1(x)h_2(x)$ where $h_1(x) = \sin^2(x)$ and $h_2(x) = \sin(x^3)$. We leave it as exercise to the reader to verify by induction that

$$g^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} h_1^{(k)}(x)h_2^{(n-k)}(x).$$

Now, we compute $h_1'(x) = 2\sin(x)\cos(x)$, $h_1''(x) = 2\left(\cos^2(x) - \sin^2(x)\right)$, $h_2'(x) = 3x^2\cos(x^3)$, $h_2''(x) = 6x\cos(x^3) - 9x^4\sin(x^3)$, and $h_2'''(x) = 6\cos(x^3) - 18x^3\sin(x^3) - 36x^3\sin(x^3) - 27x^6\cos(x^3)$. Note $h_1(0) = h_1'(0) = h_2(0) = h_2'(0) = h_2''(0) = 0$. These facts together with the above formula prove that $h^{(k)}(x) = 0$ for $k = 0, \ldots, 4$. Note $h_1''(0) = 2$, $h_2'''(0) = 6$. Thus, we get $g^{(5)} = 120$.
Applying L'Hopital's rule 5 times, we end up with

$$\lim_{x\to 0} \frac{\sin(x) - xe^{x^2} + 7x^3/6}{\sin^2(x)\sin(x^3)} = -\frac{59}{120}.$$

(b) By Taylor's theorem, we know $e^x = 1 + x + x^2/2 + E_3$ where $E_3 = \frac{x^3}{6}e^c$ for some c between 0 and x. If $|x| \le 1$, then $|c| \le 1$ and $e^c \le e$ because e^t is an increasing function of t. Since $e \le 3$, we have $|E_3| \le \frac{x^3}{2}$ as desired.

(c) Plugging in x^2 in our formula from part (b), we get that $e^{x^2} = 1 + x^2 + \frac{x^4}{2} + E_3$ where $|E_3| \le \frac{x^6}{2}$ whenever $|x| \le 1$. Integrating, we get

$$\left| \int_0^t e^{x^2} dx - \left(t + \frac{t^3}{3} + \frac{t^5}{10}\right) \right| \le \frac{t^7}{14}$$

whenever $|t| \le 1$. Note that we used the comparison theorem for integrals to bound $|\int_0^t E_3(x)dx|$.

Bonus: Give an example of an infinitely differentiable function f such that f is not identically zero but $T_n f = 0$ for all n. Here $T_n f$ denotes the nth Taylor polynomial of f centered at zero.

$\boxed{\text{Solution}}$ (4 points) Consider the function

$$f(x) = \left\{ \begin{array}{ll} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{array} \right\}.$$

Clearly f is infinitely differentiable on the interval $(-\infty, 0)$. To show that $f(x)$ is infinitely differentiable on the interval $(0, \infty)$, we show by induction on n that f is n times differentiable for $x > 0$ and $f^{(n)}(x) = p_n(\frac{1}{x})e^{-1/x^2}$ for some polynomial p_n whenever $x > 0$. . The base case $n = 0$ follows from the definition of f. For the inductive step, we assume $f^{(n)}(x) = p_n(\frac{1}{x})e^{-1/x^2}$ and we show that there exists a polynomial p_{n+1} such that $f^{(n+1)}(x) = p_{n+1}(\frac{1}{x})e^{-1/x^2}$.
To do this, we simply differentiate $f^{(n)}(x) = p_n(\frac{1}{x})e^{-1/x^2}$. By the chain rule and the product rule,

$$f^{(n+1)}(x) = -p_n'\left(\frac{1}{x}\right) \cdot \frac{1}{x^2} \cdot e^{1/x^2} - 2p_n\left(\frac{1}{x}\right) \cdot \frac{1}{x^3} \cdot e^{-1/x^2}.$$

Setting $p_{n+1}(t) = -t^2 p_n'(t) - 2t^3 p_n(t)$ completes the inductive step.
In particular, we have shown that $f(x)$ is infinitely differentiable on the interval $(0, \infty)$. It remains to show that f is infinitely differentiable at 0. We need a lemma.

Lemma: If p is a polynomial, then

$$\lim_{x \to 0} p\left(\frac{1}{x}\right) e^{-1/x^2} = 0.$$

Proof: Given $\epsilon > 0$, we must find $\delta > 0$ such that $|x| < \delta$ implies $|p\left(\frac{1}{x}\right) e^{-1/x^2}| < \epsilon$. Put $t = 1/x$, and let $p(t) = \sum_{k=0}^{n} c_k t^k$. Then $|p(t)| \leq (n+1)\max_{k=0}^{n}\{|c_k|\}|t|^n$ if $t \geq 1$. Note that $e^{t^2} \geq \sum_{k=0}^{m} \frac{t^{2k}}{k!}$ by Taylor's formula (Theorem 7.6) and the bound on the error term (Theorem 7.7) together with the fact that the derivative of e^{t^2} is never negative. If we choose $2m > n$, then we see that $e^{t^2} \geq \frac{t^{2m}}{(2m)!}$ and if $t > \frac{1}{\epsilon}(n+1)\max_{k=0}^{n}\{|c_k|\} = \frac{1}{\delta}$, we conclude that

$$\frac{e^{t^2}}{|p(t)|} > \frac{1}{\epsilon}.$$

Plugging in $t = 1/x$ and taking the inverse of the above equality, we note that

$$\left| p\left(\frac{1}{x}\right) \right| e^{-1/x^2} < \epsilon$$

whenever $x < \delta$. This proves the lemma.

Back to showing that $f^{(n)}$ is differentiable at $x = 0$ for every n. Since $f^{(n)}(x) = p_n\left(\frac{1}{x}\right) e^{-1/x^2}$, we note that

$$\lim_{x \to 0^+} \frac{f^{(n)}(x) - 0}{x} = \lim_{x \to 0^+} q_n\left(\frac{1}{x}\right) e^{-1/x^2}.$$

Here $q_n(t) = tp_n(t)$. By the lemma, the above limit is zero. Since the limit from the right is obviously zero, we conclude that $f^{(n)}$ is differentiable at zero with derivative zero for all n.
Finally, $T_n f = f^{(n)}(0)x^n/n!$ by definition. Since $f^{(n)} = 0$ for all n, we have $T_n f = 0$ for all n. However, f is not identically zero. For instance, $f(1) = 1/e$.