Square roots, and the existence of irrational numbers.

Definition. If $b^2 = a$, then we say that b is a square root of a.

A negative number has no square root (see Theorem I.20), and the number 0 has only one square root, namely 0. We shall show that a positive real number has exactly two square roots, one positive and one negative.

$\boxed{\textit{Theorem. Let } a > 0. \textit{ Then there is a number } b > 0 \textit{ such that } b^2 = a.}$

Proof. Step 1. Let x *and* y *be positive numbers. Then* $x < y$ *if and only if* $x^2 < y^2$.

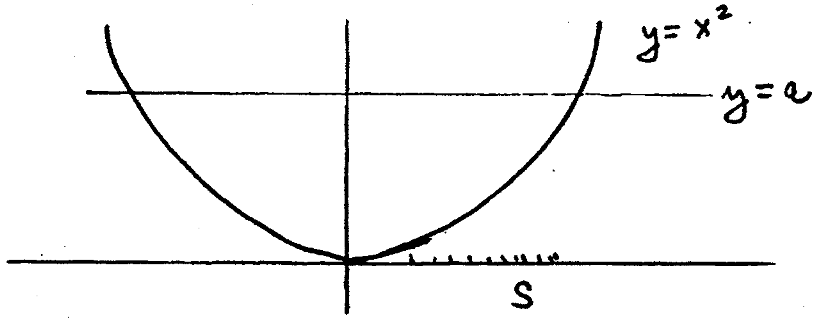
If $x < y$, we multiply both sides, first by x and then by y, to obtain the inequalities

$$x \cdot x < y \cdot x \quad \text{and} \quad y \cdot x < y \cdot y.$$

Thus $x^2 < y^2$. Conversely if $x^2 < y^2$, then it cannot be true that $x = y$ (for that would imply $x^2 = y^2$), or that $y < x$ (for that would imply, by what we just proved, that $y^2 < x^2$). Hence we must have $x < y$.

Step 2. We construct b as follows: Consider the set

$$S = \{x \mid x > 0 \quad \text{and} \quad x^2 < a\}.$$



The set S is nonempty; indeed if x is a number such that $0 < x \leq 1$ and $x < a$, then

$$x^2 < ax \leq a \cdot 1 = a,$$

so that x is in S. Furthermore, S is bounded above; indeed, $1 + a$ is an upper bound on S:

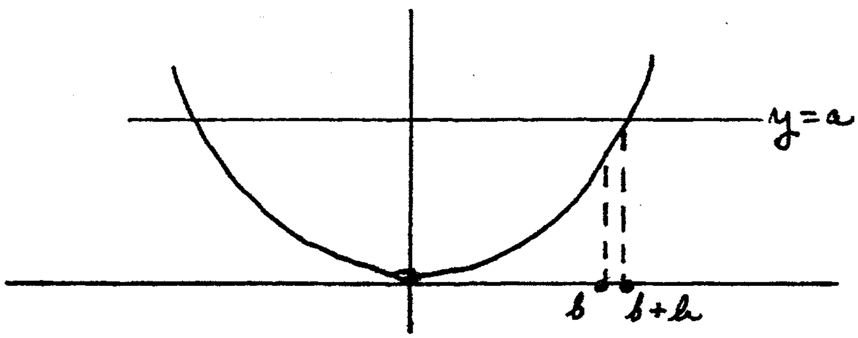
For if x is in S, then $x^2 < a$; since

$$a < 1 + 2a + a^2 = (1+a)^2,$$

it follows from Step 1 that $x < 1 + a$.

Let b denote the supremum of S; we show that $b^2 = a$. We verify this fact by showing that neither inequality $b^2 < a$ or $b^2 > a$ can hold.

Step 3. Assume first that $b^2 < a$. We shall show that there is a positive number h such that $(b+h)^2 < a$. It then follows that $b + h$ belongs to S (by definition of S), contradicting the fact that b is an upper bound for S.



To find h, we proceed as follows: The inequality $(b+h)^2 < a$ is equivalent to the inequality

$$h(2b+h) < a-b^2.$$

Now $a - b^2$ is positive; it seems reasonable that if we take h to be sufficiently small, this inequality will hold. Specifically, we first specify that $h \leq 1$; then we have

$$h(2b+h) \leq h(2b+1).$$

It is then easy to see how small h should be; if we choose $h < (a-b^2)/(2b+1)$, then

$$h(2b+1) < a - b^2$$

and we are finished.

Step 4. Now assume that $b^2 > a$. We shall show that there is a number h such that $0 < h < b$ and $(b-h)^2 > a$. It follows that $b - h$ is an upper bound for S: For if x is in S, then $a > x^2$, so that $(b-h)^2 > x^2$, whence by Step 1, $b - h > x$. This contradicts the fact that b is the least upper bound for S.

To find h, we proceed as follows: The inequality $(b-h)^2 > a$ is equivalent to the inequality

$$h(2b-h) < b^2 - a.$$

Now $b^2 - a$ is positive; it seems reasonable that if h is sufficiently small, this inequality will hold. Our first requirement is that $0 < h < b$. Then we note that $h(2b-h) = 2hb - h^2 < 2hb$. It is now easy to see how small h should be; if we choose $h < (b^2-a)/2b$, then

$$2hb < b^2 - a$$

and we are finished. □

Corollary. If $a > 0$, *then* a *has exactly two square roots.*

We denote the positive square root of a by $\sqrt{a}$.

Proof. Let $b > 0$ and $b^2 = a$. Then $(-b)^2 = a$. Thus a has at least two square roots, b and $-b$. Conversely, if c is any square root of a, then $c^2 = a$, whence

$$(b+c)(b-c) = b^2 - c^2 = 0.$$

It follows that $c = -b$ or $c = b$. □

We now demonstrate the existence of irrational numbers.

Theorem. Let a *be a positive integer; let* $b = \sqrt{a}$. *Then either* b *is a positive integer or* b *is irrational.*

Proof. Suppose that $b = \sqrt{a}$ and b is a rational number that is not an integer. We derive a contradiction.

Let us write $b = m/n$, where m and n are positive integers and n is as small as possible. (I.e., we choose n to be the smallest positive integer such that nb is an integer, and we set $m = nb$.)

Choose q to be the unique integer such that

$$q < m/n < q+1.$$

Then

$$qn < m < qn + n, \text{ or}$$

$$(*) \qquad 0 < m - qn < n.$$

We compute as follows:

$$(m/n)^2 = b^2 = a,$$
$$m^2 = n^2 a$$
$$m(m-qn) = n(na-qm).$$

Then using (*), we can write

$$b = \frac{m}{n} = \frac{na-qm}{m-qn}.$$

This equation expresses b as a ratio of positive integers; and by (*) the denominator is less than n. Thus we reach a contradiction. □

Corollary. $\sqrt{2}$ *is irrational.*

Proof. Let $b = \sqrt{2}$. Then b cannot be an integer, for the square of 1 equals 1 while the square of any integer greater than 1 is at least 4. It follows that b is irrational. □

The same proof shows that the number $\sqrt{n}$ is irrational whenever n is a positive integer less than 100 that is not one of the integers 1, 4, 9, 16, 25, 36, 49, 64, or 81.

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