

1.17 The Riemann condition.

The most useful criterion for determining whether  $f$  is integrable on  $[a,b]$  is given in the following theorem. It is called the Riemann condition for existence of the integral.

Theorem 1. Suppose  $f$  is defined on  $[a,b]$ . Then  $f$  is integrable on  $[a,b]$  if and only if given any  $\epsilon > 0$ , there exist, correspondingly, step functions  $s$  and  $t$ , with  $s \leq f \leq t$  on  $[a,b]$ , such that

$$\int_a^b t - \int_a^b s < \epsilon.$$

Proof. We know from Theorem 1.9 of the text (p. 74) that

$$\int_a^b s \leq \underline{I} \leq \bar{I} \leq \int_a^b t,$$

provided  $s$  and  $t$  are step functions such that  $s \leq f \leq t$ . Suppose that the condition of the theorem holds. Given  $\epsilon > 0$ , choose  $s \leq f \leq t$  so that  $\int_a^b t - \int_a^b s < \epsilon$ . It follows that  $\bar{I} - \underline{I} < \epsilon$ . Because this latter inequality holds for every positive  $\epsilon$ , it must be true that  $\bar{I} = \underline{I}$ . Hence  $f$  is integrable.

Conversely, suppose  $f$  is integrable. Given  $\epsilon > 0$ , choose a step function  $s$  with  $s \leq f$ , such that  $\int_a^b s$  is within  $\epsilon/2$  of  $\underline{I}$ . This we can do because  $\underline{I}$  is the supremum of the set of such numbers  $\int_a^b s$ . Similarly choose  $t \geq f$  so

that  $\int_a^b t$  is within  $\epsilon/2$  of  $\bar{I}$ . Now because  $f$  is integrable,  $\underline{I} = \bar{I}$ . Therefore,  $\int_a^b t$  and  $\int_a^b s$  are within  $\epsilon$  of each other.  $\square$

We now obtain a slight strengthening of the preceding theorem:

Theorem 2. Suppose  $f$  is defined on  $[a,b]$ . Let  $A$  be a number. Suppose that given  $\epsilon > 0$ , there exist step functions  $s$  and  $t$  with  $s \leq f \leq t$ , such that

$$\int_a^b t - \int_a^b s < \epsilon, \text{ and}$$

$$\int_a^b s < A < \int_a^b t.$$

Then  $\int_a^b f$  exists and equals  $A$ .

Proof. We know that  $\int_a^b f$  exists, by the preceding theorem. Given  $\epsilon > 0$ , choose  $s$  and  $t$  satisfying the hypotheses of the corollary. Then we have

$$\int_a^b s < A < \int_a^b t$$

by hypothesis, and

$$\int_a^b s < \int_a^b f < \int_a^b t$$

by definition of the integral. Because  $\int_a^b t - \int_a^b s < \epsilon$ , it follows that

$$|A - \int_a^b f| < \varepsilon.$$

Since the latter inequality holds for every  $\varepsilon > 0$ , we must have  $A = \int_a^b f$ .  $\square$

The calculation of  $\int x^p$ .

Let  $p$  be a positive integer. We seek to show that the function  $f(x) = x^p$  is integrable, and to obtain a formula for it, using the definition of the integral. The proof proceeds by applying the Riemann condition. It involves some hard work, as we shall see.

We need the following lemma, which is easily proved by induction. (See the Exercises of Section A.)

Lemma 3. Let  $a$  and  $a + h$  be positive real numbers.  
If  $m$  is a positive integer, we have

$$(a+h)^m \geq a^m + ma^{m-1}h.$$

Now we prove our desired integration formula. First we consider a special case:

Theorem 4. Let  $b$  be a positive real number. Then

$$\int_0^b x^p dx = \frac{b^{p+1}}{p+1} .$$

Proof. The theorem is proved by applying the Riemann condition, as expressed in Theorem 2 preceding. We show that given  $\epsilon > 0$ , there are step functions  $s$  and  $t$  such that

- (i)  $s \leq f \leq t$  on  $[0, b]$ ,
- (ii)  $\int_0^b t - \int_0^b s < \epsilon$ , and
- (iii)  $\int_0^b s \leq b^{p+1}/(p+1) \leq \int_0^b t$ .

This will prove our result.

Let  $f(x) = x^p$ . To define  $s$  and  $t$ , we begin by partitioning  $[0, b]$  into  $n$  equal intervals. That is, we consider the partition

$$x_0 = 0, x_1 = b/n, \dots, x_k = kb/n, \dots, x_n = nb/n = b.$$

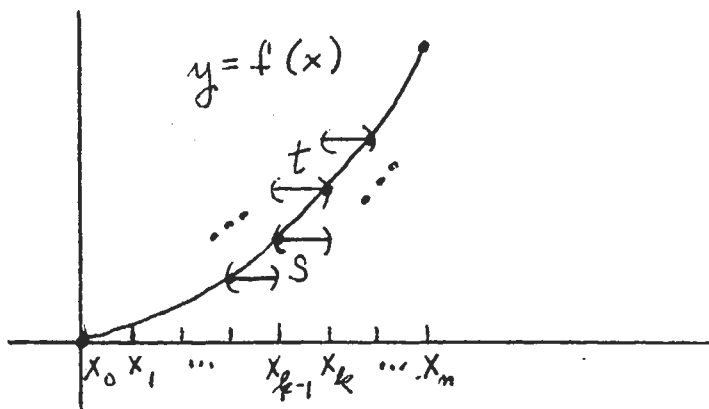
It is not obvious how  $n$  should be chosen. In fact, given  $\epsilon > 0$ , we shall choose  $n > b^{p+1}/\epsilon$ ; we will see later that this is the "right" choice.

Now  $f(x)$  is strictly increasing on  $[0,b]$ ; that is, if  $0 \leq x_1 < x_2$ , then  $f(x_1) < f(x_2)$ . (This result is easily proved by induction on  $p$ .) Hence it is easy to define  $s$  and  $t$ . We let

$$s(x) = f(x_{k-1}) \quad \text{for } x_{k-1} < x < x_k, \quad \text{and}$$

$$t(x) = f(x_k) \quad \text{for } x_{k-1} < x < x_k.$$

Then  $s \leq f \leq t$  on the interval  $(x_{k-1}, x_k)$ .



These equations define  $s$  and  $t$  except at the partition points. It doesn't matter how we define them at the partition points, so long as  $s \leq f \leq t$  holds. Suppose we let  $s(x) = 0$  and  $t(x) = b^p$  at the partition points. Then  $s \leq f \leq t$  on the entire interval  $[0,b]$ . Thus (i) holds.

Let us compute  $\int_0^b t - \int_0^b s$  and show (ii) holds. Now the value of  $s(x)$  on  $(x_{k-1}, x_k)$  is given by

$$s_k = f(x_{k-1}) = (x_{k-1})^p = ((k-1)b/n)^p.$$

Similarly, the value of  $t(x)$  on this same interval is

$$t_k = f(x_k) = (x_k)^p = (kb/n)^p.$$

Furthermore,  $x_k - x_{k-1} = b/n$ . We compute

$$\begin{aligned} \int_0^b s &= \sum_{k=1}^n s_k \cdot (x_k - x_{k-1}) \\ &= \sum_{k=1}^n s_k \cdot (b/n) \\ &= \left[ 0^p + \left(\frac{b}{n}\right)^p + \left(\frac{2b}{n}\right)^p + \dots + \left(\frac{(n-1)b}{n}\right)^p \right] \left(\frac{b}{n}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} \int_0^b t &= \sum_{k=1}^n t_k \cdot (b/n) \\ &= \left[ \left(\frac{b}{n}\right)^p + \left(\frac{2b}{n}\right)^p + \dots + \left(\frac{nb}{n}\right)^p \right] \left(\frac{b}{n}\right). \end{aligned}$$

Subtracting these equations, we obtain

$$\int_0^b t - \int_0^b s = \left(\frac{nb}{n}\right)^p \left(\frac{b}{n}\right) = \frac{b^{p+1}}{n}.$$

Since we (cleverly!) chose  $n$  so that  $n > b^{p+1}/\epsilon$ , it follows that, as desired.

$$\int_0^b t - \int_0^b s < \epsilon.$$

To check (iii) requires some work. Plugging the preceding computations into the desired inequalities

$$\int_0^b s \leq b^{p+1}/(p+1) \leq \int_0^b t,$$

we obtain the inequalities

$$b^{p+1} \frac{[0^p + 1^p + 2^p + \dots + (n-1)^p]}{n^{p+1}} \leq \frac{b^{p+1}}{p+1} \leq b^{p+1} \frac{[1^p + 2^p + \dots + n^p]}{n^{p+1}}.$$

Simplifying further, we have

$$(*) \quad 0^p + 1^p + \dots + (n-1)^p \leq \frac{n^{p+1}}{p+1} \leq 1^p + 2^p + \dots + n^p.$$

These are the inequalities we must prove.

We proceed by induction on  $n$ , holding  $p$  fixed. Both inequalities are trivial when  $n = 1$ . We assume them true for  $n$ , and verify their correctness for  $n + 1$ .

Let us begin with the left inequality in (\*). We add  $n^p$  to both sides to obtain

$$0^p + 1^p + \dots + (n-1)^p + n^p \leq \frac{n^{p+1}}{p+1} + n^p.$$

If we can show that

$$\frac{n^{p+1}}{p+1} + n^p \leq \frac{(n+1)^{p+1}}{p+1},$$

we are through; for the left inequality in (\*) then holds for  $n + 1$  and the induction step is verified. But this latter

inequality follows at once from Lemma 3. If we set  $a = n$  and  $h = 1$  and  $m = p+1$ , this lemma takes the form

$$(n+1)^{p+1} \geq n^{p+1} + (p+1)n^p,$$

which (if we divide through by  $p+1$ ) is exactly what we want.

Now we consider the right inequality in (\*). Adding  $(n+1)^p$  to both sides, we obtain the inequality

$$\frac{n^{p+1}}{p+1} + (n+1)^p \leq 1^p + 2^p + \dots + n^p + (n+1)^p.$$

If we can show that

$$\frac{(n+1)^{p+1}}{p+1} \leq \frac{n^{p+1}}{p+1} + (n+1)^p,$$

then we are through; the right inequality of (\*) then holds for  $n + 1$ , and the induction step is verified. Once again, we use Lemma 3. If we set  $a = n+1$  and  $h = -1$  and  $m = p+1$  in that lemma, we obtain

$$n^{p+1} \geq (n+1)^{p+1} + (p+1)(n+1)^p(-1),$$

which gives our desired inequality.  $\square$

Using the basic properties of the integral (proved in D of these notes), we now derive the general integration formula for  $x^p$ .



Theorem 5. Let  $p$  be a positive integer. Then for all  
 $a$  and  $b$ ,

$$\int_a^b x^p dx = \frac{b^{p+1} - a^{p+1}}{p+1}.$$

Proof. Step 1. We first verify that the given formula holds when  $a = 0$ . That is, we show that for all  $b$ ,

$$\int_0^b x^p dx = b^{p+1}/(p+1).$$

The case  $b > 0$  was proved in Theorem 4. The case  $b = 0$  is trivial. Consider the case  $b < 0$ ; let  $b = -c$ , where  $c > 0$ . Applying basic properties of the integral, we compute

$$\begin{aligned} \int_0^c x^p dx &= \int_{-c}^0 (-x)^p dx && \text{by the reflection property,} \\ &= \int_b^0 (-1)^p x^p dx && \text{by laws of exponents,} \\ &= (-1)^p \int_b^0 x^p dx && \text{by the linearity property,} \\ &= (-1)^{p+1} \int_0^b x^p dx && \text{by our convention.} \end{aligned}$$

On the other hand, Theorem 4 implies that

$$\begin{aligned} \int_0^c x^p dx &= c^{p+1}/(p+1) = (-b)^{p+1}/(p+1) \\ &= (-1)^{p+1} b^{p+1}/(p+1). \end{aligned}$$

Comparing these two computations gives us our desired formula.

Step 2. The theorem now follows. We have

$$\begin{aligned}
 \int_a^b x^p dx &= \int_a^0 x^p dx + \int_0^b x^p dx && \text{by the additivity property,} \\
 &= \int_0^b x^p dx - \int_0^a x^p dx && \text{by our convention,} \\
 &= \frac{b^{p+1}}{p+1} - \frac{a^{p+1}}{p+1} && \text{by Step 1 of this proof. } \square
 \end{aligned}$$

NOTE: Let us introduce the notation

$$f(x) \Big|_a^b = f(b) - f(a).$$

With this notation, the preceding theorem can be written in the form

$$\int_a^b x^p dx = \frac{x^{p+1}}{p+1} \Big|_a^b.$$

NOTE: The preceding theorem, along with the linearity property of the integral, now enables us to compute the integral of any polynomial function. We simply "integrate term-by-term."

For example, one has the following computation:

$$\begin{aligned}\int_1^2 (x^3 - 3x + 5) dx &= \int_1^2 x^3 dx - 3 \int_1^2 x dx + 5 \int_1^2 1 dx \\ &= \left. \frac{x^4}{4} \right|_1^2 - 3 \left. \frac{x^2}{2} \right|_1^2 + 5 \left. x \right|_1^2 \\ &= \left( 4 - \frac{1}{4} \right) - 3 \left( 2 - \frac{1}{2} \right) + 5(2 - 1) \\ &= \frac{15}{4} - \frac{9}{2} + 5 = \frac{17}{4} .\end{aligned}$$

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