## L'Hopital's rule for 0/0

Theorem. Suppose $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$. If

$$\frac{f'(x)}{g'(x)} \to L \quad \text{as} \quad x \to a,$$

then also $f(x)/g(x) \to L$ as $x \to a$.

This result holds whether $a$ and $L$ are finite or infinite, and it also holds if the limits are one-sided.

Proof. The proof when $a$ is finite is that given on p. 295 of the text. The crucial step is to use Cauchy's mean-value theorem to prove that $g(x) \neq 0$ for $x$ near $a$, and that

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

for some $c$ between $a$ and $x$. It follows that if $f'(x)/g'(x)$ approaches $L$ as $x \to a$, then $f(x)/g(x)$ must approach $L$ also. In the text, it is assumed that $L$ is finite. But it does really not matter whether $L$ is finite or infinite; precisely the same proof applies.

The proof in the case $a = +\infty$ is given on p. 298 of the text. Again, it is assumed that $L$ is finite, but that doesn't matter; if $L$ is $\pm\infty$ precisely the same proof applies. □

Remark. L'Hopital's rule also works if $f(x)$ and $g(x)$ both approach $\infty$ instead of 0. But the proof is more complicated. We shall give a proof shortly. The only cases of interest to us concern the logarithm and the exponential. For these functions, a direct proof is given on p. 301 of the text. Alternatively, they may be treated by using L'Hopital's rule for the case $\infty/\infty$, as we shall see.

The behavior of log and exp that we are concerned with is stated in the following theorem:

Theorem. As $x \to +\infty$, both $\log x$ and $e^x$ approach $+\infty$. But $\log x$ approaches $\infty$ more slowly than any positive power of $x$, and $e^x$ approaches $\infty$ more rapidly than any positive power of $x$; the same holds for any positive powers of $\log x$ and $e^x$. More precisely, if $a$ and $b$ are positive real numbers, then

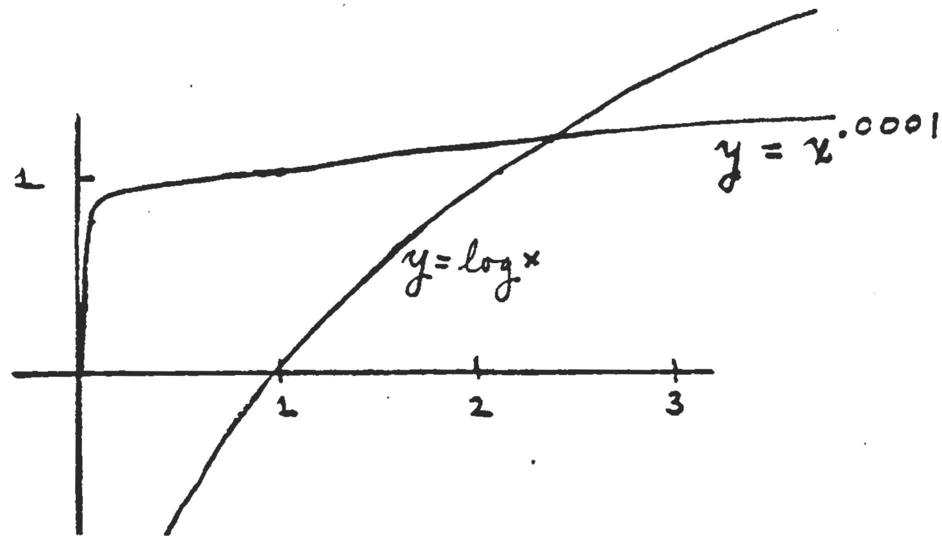
$$\lim_{x\to+\infty} \frac{(\log x)^b}{x^a} = 0 \quad \text{and} \quad \lim_{x\to+\infty} \frac{(e^x)^b}{x^a} = +\infty$$

Corollary. The function $\log x$ goes to $-\infty$ very slowly as $x$ goes to 0. More precisely, if $a$ is a positive real number, then

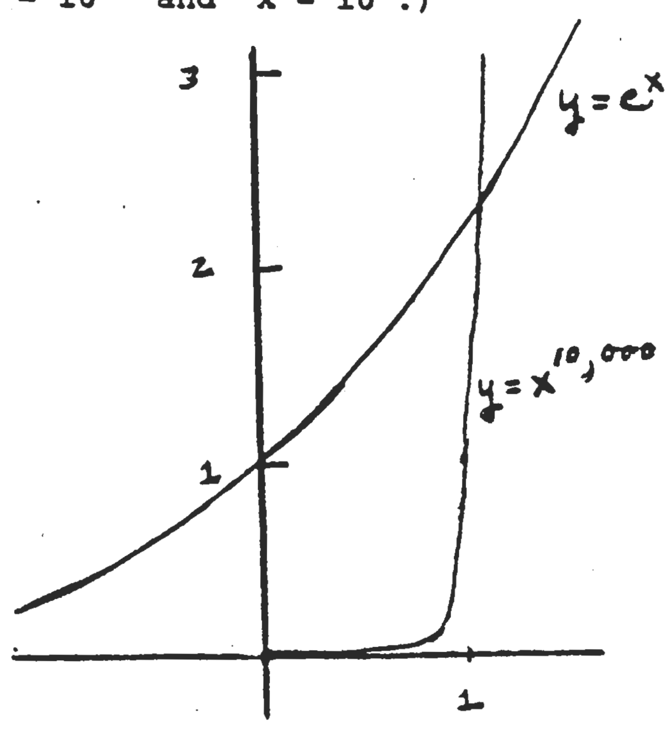
$$\lim_{x\to 0+} x^a \log x = 0 .$$

What does this theorem mean? Note that for any function $f(x)$ that goes to $\infty$ as $x$ goes to $\infty$, a positive power of $f(x)$, say $[f(x)]^a$, goes to $\infty$ even more rapidly if the power $a$ is large, and to goes to $\infty$ more slowly if $a$ is small. This theorem says that no matter how high a power $b$ you raise $\log x$ to, and how small a power $a$ you raise $x$ to, the power of $\log x$ will still go to $\infty$ more slowly than the power of $x$. Similarly, any power of $e^x$, no matter how small, will go to $\infty$ faster than any power of $x$, no matter how large.

For example, even though for small values of $x$, the graphs of the functions $\log x$ and $x^{.0001}$ appear as in the accompanying figure, it is still true that eventually the function $f(x) = x^{.0001}$ becomes much larger than $\log x$.




Similar graphs for the functions $x^{10,000}$ and $e^x$ can be obtained by exchanging the axes in this figure. Although $x^{10,000}$ shoots up very rapidly to begin with, eventually $e^x$ becomes much larger than $x^{10,000}$. (In fact, these curves cross again between $x = 10^5$ and $x = 10^6$.)



L'Hopital's rule for $\infty/\infty$.

Theorem. Suppose $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$. If

$$\frac{f'(x)}{g'(x)} \longrightarrow L \text{ as } x \longrightarrow a,$$

then also $f(x)/g(x) \longrightarrow L$ as $x \longrightarrow a$.

This result holds whether a and L are finite or infinite, and it also holds if the limits are one–sided.

Proof. Case 1. We prove the theorem first in the case where a is finite and $x \longrightarrow a+$.

The hypotheses of the theorem imply that f and g are defined and positive on some interval of the form $(a,b]$, and that $f'$ and $g'$ exist and $g' \neq 0$ on some such interval.

Let $x_0$ be a fixed point of this interval. (We shall specify how to choose $x_0$ later.) Then let x be a point of this interval that is very close to a. Just how close will be determined later. For now we merely require that $a < x < x_0$ and that $f(x) > f(x_0)$ and $g(x) > g(x_0)$. (Since f and g go to $\infty$ as $x \to a+$, these inequalities hold if x is close enough to a.) Then we compute.

Let us apply the Cauchy mean-value theorem to the interval $[x,x_0]$. We conclude that there is a c with $x < c < x_0$ such that

$$f'(c)[g(x_0)-g(x)] = g'(c)[f(x_0)-f(x)]$$

or

$$f'(c)g(x)[g(x_0)/g(x) -1] = g'(c)f(x)[ f(x_0)/f(x)-1]$$

or

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}\left[\frac{(g(x_0)/g(x)) - 1}{(f(x_0)/f(x)) - 1}\right].$$

For convenience, let $\lambda(x)$ denote the expression in brackets; then

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}\lambda(x).$$

Note that $\lambda(x) \to 1$ as $x \to a+$.

Now we verify the theorem in the case where L is finite. By choosing $x_0$ close to a, we can ensure that $f'(c)/g'(c)$ is close to L (since $a < c < x_0$); then we can make $\lambda(x)$ close to 1 by requiring that x be very close to a. Then $f(x)/g(x)$ will be close to L. The only question is: how close is "close enough"? Let us set

$$\epsilon_1 = |(f'(c)/g'(c))-L| \quad \text{and} \quad \epsilon_2 = |\lambda(x)-1|.$$

Then

$$(*) \qquad \left|\frac{f'(c)}{g'(c)}\cdot\lambda(x)-L\right| = |(L\pm\epsilon_1)(1\pm\epsilon_2)-L| \le |\epsilon_1| + |L\epsilon_2| + |\epsilon_1\epsilon_2|.$$

This inequality tells us how to proceed. Suppose $0 < \epsilon < 1$. First, we choose $x_0$ so that for all c with $a < c < x_0$, we have $\epsilon_1 < \epsilon/3$. Now $x_0$ is fixed. Then choose $\delta > 0$ so that for $a < x < a+\delta$, we have $g(x) > g(x_0)$ and $f(x) > f(x_0)$ and

$$|\lambda(x)-1| = \epsilon_2 < \epsilon/3(1+|L|).$$

Then for $a < x < a+\delta$, inequality (*) tells us that

$$\left|\frac{f(x)}{g(x)} - L\right| = \left|\frac{f(c')}{g(c')}\,\lambda(x) - L\right| \le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon^2}{9} < \epsilon, \text{ as desired.}$$

Finally, we consider the case where L is infinite. Given $M > 0$, we want to show that $f(x)/g(x) > M$ for x close to a. First, choose $x_0$ so that for all c with $a < c < x_0$, we have $f'(c)/g'(c) > 2M$. Then choose $\delta$ so that for $a < x < a+\delta$, we have $g(x) > g(x_0)$ and $f(x) > f(x_0)$ and $\lambda(x) > 1/2$. It follows that, for $a < x < a+\delta$,

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}\lambda(x) > 2M\cdot\frac{1}{2} = M.$$

We have now proved the rule in the case $x \to a+$. The case $x \to a-$ follows readily, as we now show. Note that as x approaches a from the left, $u = a-x$ approaches 0 from the right. Then

$$\begin{aligned}\lim_{x\to a-} (f(x)/g(x)) &= \lim_{u\to 0+} f(a-u)/g(a-u) \\ &= \lim_{u\to 0+} (-1)f'(a-u)/(-1)g'(a-u) \\ &= \lim_{x\to a-} f'(x)/g'(x),\end{aligned}$$

if the latter limit exists.

The case $x \to a$, with a finite, follows from the two cases $x \to a+$ and $x \to a-$.

Finally, the case $x \to \infty$ follows from the computation

$$\begin{aligned}\lim_{x\to\infty} f(x)/g(x) &= \lim_{t\to 0+} f(1/t)/g(1/t) \\ &= \lim_{t\to 0+} (-1/t^2)f'(1/t)/(-1/t^2)g'(1/t) \\ &= \lim_{x\to\infty} f'(x)/g'(x),\end{aligned}$$

if the latter limit exists. □

The behavior of log and exp

We now derive the theorem on p. P.2 from L'Hopital's rule. Consider first the log function. Given $c > 0$, we compute

$$\begin{aligned}\lim_{x\to\infty}(\log x)/x^c &= \lim_{x\to\infty} x^{-1}/cx^{c-1} \quad \text{by L'Hopital's rule} \\ &= \lim_{x\to\infty} 1/cx^c = 0.\end{aligned}$$

Then we set $c = b/a$ and compute

$$\lim_{x\to\infty} (\log x)^a/x^b = \lim_{x\to\infty} [\log x/x^c]^a = 0,$$

as desired.

Now we consider the exp function. Given $c > 0$, we compute

$$\begin{aligned}\lim_{x\to\infty} e^{cx}/x &= \lim_{x\to\infty} ce^{cx}/1 \quad \text{by L'Hopital's rule} \\ &= \infty.\end{aligned}$$

Then we set $c = a/b$ and compute

$$\lim_{x\to\infty} (e^x)^a/x^b = \lim_{x\to\infty}[e^{cx}/x]^b = \infty,$$

as desired.

Finally, we note that

$$\lim_{x \to 0+} x^a \log x = \lim_{t \to \infty} (1/t^a)\log(1/t)$$
$$= \lim_{t \to \infty} \frac{-\log t}{t^a} = 0,$$

as desired□

Example. Although

$$\lim_{x \to \infty} \frac{x+\sin x}{x}$$

assumes the indeterminate form $\infty/\infty$, L'Hopital's rule does not apply, since the function $(1+\cos x)/1$ oscillates rather than approaches a limit as $x \to \infty$. However,

$$\frac{x+\sin x}{x} = 1 + \frac{\sin x}{x},$$

which approaches 1 because $|\sin x|/x \leq 1/x$ for $x > 0$.

This example shows that the converse of L'Hopital's rule is not true. For this is a case where $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$, and $f(x)/g(x)$ approaches a limit, even though $f'(x)/g'(x)$ does not approach a limit.

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