A family of non-analytic functions.

Let $m \geq 0$ be any nonnegative integer. Define

$$f_m(x) = \begin{cases} \dfrac{e^{-1/x^2}}{x^m} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

We will show that each of the functions $f_m(x)$ has continuous derivatives of all orders, for all x. We also show that none of them is analytic near 0; that is, none of them equals a power series of the form $\Sigma\, a_n x^n$ in an interval about 0.

Theorem 1. (a) *The function $f_m(x)$ is continuous for all x.*

(b) *Furthermore, $f_m'(x)$ exists for all x and satisfies the equation*

$$f_m'(x) = -mf_{m+1}(x) + 2f_{m+3}(x).$$

Proof. (a) The general theorem about composites of continuous functions shows that $f_m(x)$ is continuous when $x \neq 0$. To prove continuity at $x = 0$, we must show that

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^m} = 0.$$

The substitution $\mu = 1/x^2$ simplifies the calculation. We have

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^m} = \lim_{\mu\to\infty} \frac{e^{-\mu}}{1/\mu^{m/2}} = \lim_{\mu\to\infty} \frac{\mu^{m/2}}{e^{\mu}}\ .$$

This limit is zero because e^{μ} approaches infinity faster than any power of μ, as $\mu \longrightarrow \infty$.

(b) We check differentiability. If $x \neq 0$, we calculate directly:

$$f_m'(x) = D\left(\frac{1}{x^m} e^{-1/x^2}\right) = \frac{-m}{x^{m+1}} e^{-1/x^2} + \frac{1}{x^m} e^{-1/x^2}\left(\frac{2}{x^3}\right)$$

$$= -mf_{m+1}(x) + 2f_{m+3}(x).$$

To show the derivative exists at $x = 0$, we apply the definition of the derivative:

$$f_m'(0) = \lim_{h\to 0} \frac{f_m(0+h) - f_m(0)}{h}$$

$$= \lim_{h\to 0} \frac{(e^{-1/h^2}/h^m) - 0}{h} = \lim_{h\to 0} \frac{e^{-1/h^2}}{h^{m+1}}$$

This limit is zero, by part (a). Therefore, the derivative exists at $x = 0$ and equals 0. Thus the formula

$$f_m'(x) = -mf_{m+1}(x) + 2f_{m+3}(x)$$

holds when $x = 0$.

<u>Theorem 2</u>. <u>The function $f_m(x)$ has continuous derivatives of all orders, for all x, but $f_m(x)$ does not equal a power series $\Sigma\, a_n x^n$ on any interval about 0.</u>

<u>Proof</u>. We know that each function $f_m(x)$ is differentiable, for all x. The equation

$$f'_m(x) = -mf_{m+1}(x) + 2f_{m+3}(x)$$

shows us that $f'_m(x)$ is differentiable, for each x. This is the same as saying that derivative $f''_m(x)$ exists for all x.

In general, we proceed by induction. Suppose we are given that the $n^{\underline{th}}$ derivative of each function $f_m(x)$ exists, for all x. Then the preceding equation shows that the $n^{\underline{th}}$ derivative of the function $f'_m(x)$ also exists, for all x. This is the same as saying that the $(n+1)^{\underline{st}}$ derivative of $f_m(x)$ exists.

It follows that the $n^{\underline{th}}$ derivative of $f_m(x)$ exists, for all x and all n. And of course it is continuous because the $(n+1)^{\underline{st}}$ derivative exists.

Now we suppose $f_m(x) = \Sigma a_n x^n$ on some non-trivial interval about $x = 0$, and derive a contradiction. If $f_m(x)$ equals this power series, then the coefficients a_n must satisfy the equations

$$a_n = \frac{f_m^{(n)}(0)}{n!}$$

for all n. We know that $f_m(x)$ vanishes when $x = 0$. Using the equation

$$f'_m(x) = -mf_{m+1}(x) + 2f_{m+3}(x)$$

repeatedly, we see that all the derivatives of $f_m(x)$ also vanish at $x = 0$. Therefore $a_n = 0$ for all n, so $f_m(x)$

is identically zero in some interval about $x = 0$. But this is not true; indeed the function $f_m(x)$ vanishes <u>only</u> for $x = 0$.