Implicit Differentiation and the Second Derivative

Calculate y'' using implicit differentiation; simplify as much as possible.

$$x^2 + 4y^2 = 1$$

Solution

As with the direct method, we calculate the second derivative by differentiating twice. With implicit differentiation this leaves us with a formula for y'' that involves y and y', and simplifying is a serious consideration.

Recall that to take the derivative of $4y^2$ with respect to x we first take the derivative with respect to y and then multiply by y'; this is the "derivative of the inside function" mentioned in the chain rule, while the derivative of the outside function is $8y$.

So, differentiating both sides of:

$$x^2 + 4y^2 = 1$$

gives us:

$$2x + 8yy' = 0.$$

We're now faced with a choice. We could immediately perform implicit differentiation again, or we could solve for y' and differentiate again.

If we differentiate again we get:

$$2 + 8yy'' + 8(y')^2 = 0.$$

In order to solve this for y'' we will need to solve the earlier equation for y', so it seems most efficient to solve for y' before taking a second derivative.

$$\begin{aligned} 2x + 8yy' &= 0 \\ 8yy' &= -2x \\ y' &= \frac{-2x}{8y} \\ y' &= \frac{-x}{4y} \end{aligned}$$

Differentiating both sides of this expression (using the quotient rule and implicit differentiation), we get:

$$\begin{aligned} y'' &= \frac{(-1)4y - (-x) \cdot 4y'}{(4y)^2} \\ &= \frac{-4y + 4xy'}{16y^2} \\ y'' &= \frac{-y + xy'}{4y^2} \end{aligned}$$

We now substitute $\frac{-x}{4y}$ for y':

$$\begin{aligned} y'' &= \frac{-y + xy'}{4y^2} \\ &= \frac{-y + x\frac{-x}{4y}}{4y^2} \\ &= \frac{x\frac{-x}{4y} - y}{4y^2} \cdot \frac{4y}{4y} \\ &= \frac{-x^2 - 4y^2}{16y^3} \\ y'' &= -\frac{1}{16y^3} \end{aligned}$$

(Don't forget to use the relation $x^2 + 4y^2 = 1$ at the end!)

How can we check our work? If we recognize $x^2 + 4y^2 = 1$ as the equation of an ellipse, we can test our equation $y' = -x/4y$ at the points $(0, 1/2)$ and $(1, 0)$. At $(0, 1/2)$, $y' = -x/4y = 0$ which agrees with the fact that the tangent line to the ellipse is horizontal at that point. At $(1, 0)$ y' is undefined, which agrees with the fact that the tangent line to the ellipse at $(1, 0)$ is vertical.

Once we have learned how the value of the second derivative is related to the shape of the graph, we can do a similar test of our expression for y''.