Comparing Linear Approximations to Calculator Computations

In lecture, we explored linear approximations to common functions at the point $x = 0$. In this worked example, we use the approximations to calculate values of the sine function near $x = 0$ and compare the answers to those on a scientific calculator.

Find the linear approximation to $\sin(x)$ at the point $x = 0$ and use your answer to approximate the values of $\sin(.01), \sin(.1)$ and $\sin(1)$. Check your answer on a calculator.

Solution:

Recall that linear approximation to a function $f(x)$ at a point $x = a$ just means that we use the tangent line $T(x)$ to $f(x)$ at $x = a$ to approximate the function. In the case where $f(x) = \sin(x)$, we saw from lecture that the tangent line $T(x)$ at $x = 0$ was given by:

$$\begin{aligned} T(x) = f'(0)(x-0) + f(0) &= \cos(0)(x-0) + \sin(0) \\ &= x. \end{aligned}$$

In short, we write $\sin(x) \approx x$ when $x \approx 0$. (Try drawing a picture of the sine curve and its tangent line at $x = 0$ to illustrate this.) So we would approximate the values of sine above as follows:

$$\begin{aligned} \sin(.01) &\approx .01 \\ \sin(.1) &\approx .1 \\ \sin(1) &\approx 1 \end{aligned}$$

Our expectation is that the closer we choose our estimation point to $x = 0$ (where the tangent line *meets* the function), the better our approximation. And indeed the calculator confirms:

$$\begin{aligned} \sin(.01) &= .00999983333... \\ \sin(.1) &= .099883341... \\ \sin(1) &= 0.84147098... \end{aligned}$$

where we've only recorded the first few digits of the decimal expansion. (Be careful to set your caculator to "radians" not "degrees" in computing these.) So $\sin(.01)$ differs from our approximation .01 by less than .0000002, a very accurate approximation! On the other hand $\sin(1)$ differs from 1 by more than .15. Note the approximation must get worse and worse in our example, as $\sin(x)$ is always bounded between -1 and 1, while x continues to grow without bound.

What if we wanted to approximate sin(1000)?

We've seen that the linear approximation at $x = 0$ would give the answer as 1000. Clearly this is far too big. Instead we should take the tangent line at a point closer to $x = 1000$ where we know the value of the sine function. The general answer for the tangent line to $f(x) = \sin(x)$ at $x = a$ is:

$$y = f'(a)(x-a) + f(a) = \cos(a)(x-a) + \sin(a)$$

so we should choose a for which we know the values of $\cos(a)$ and $\sin(a)$. We might choose $a = 318\pi$, which is around 999. Alternatively, we could use the fact that sine is periodic, with period 2π, so that $\sin(1000) = \sin(1000 - 318\pi)$, and then use the approximation at $x = 0$ for the latter value $\sin(1000 - 318\pi) = \sin(.97353...)$. We can do better still if we use fractional multiples of π and apply trigonometric identities.