Graph Sketching Example

Example: Sketch the graph of $f(x) = \frac{x+1}{x+2}$.

What does the derivative of this function tell us about its graph? To save time, I'll tell you that

$$f'(x) = \frac{1}{(x+2)^2}.$$

The special thing about this example is that $f'(x) \neq 0$, i.e. there are no values of x_0 for which $f(x_0) = 0$ and there are no critical points. If you've been trained that the first step in sketching graphs is to find out where the derivative is zero, you might be tempted to give up at this point. But we're not going to give up — can you think of anything you'd like to try here?

Student: Find values of x where f is undefined.

That's a sophisticated way of putting the point that I want to make, which is that we should go back to our precalculus skills and just plot points.

It turns out that the most important point to plot is the one that's not there: the one with x coordinate -2. This is what was just suggested — this is a point where the function is not defined.

If we try to compute $f(-2)$ we find that we can't because that makes the denominator 0. Instead we'll look at the limit as x approaches -2 from the left and from the right. We'll write a^+ for "a number x that is a little big bigger than a" to make these limits easier to talk about.

$$\lim_{x \to -2^+} f(x) = f(-2^+) = \frac{-2+1}{(-2)^+ + 2} = -10^+ = -\infty.$$

We see that the limit comes down to dividing -1 by a very small positive number; the result of this division is a very large negative number. From the other direction we get:

$$\lim_{x \to -2^-} f(x) = f(-2^-) = \frac{-2+1}{(-2)^- + 2} = -10^- = +\infty$$

This tells us what the graph of the function is doing near $x = 2$. We could evaluate the function at any point; this was the most interesting point.

Our next step is to consider the "ends" of the graph, where $x \to +\infty$ and $x \to -\infty$. This tells us what happens to the left and right of the "screen" of our graph, just as the previous calculation told us what was happening above and below.

A simple algebraic trick makes it easy to see what will happen to the value of $f(x)$ as $x \to \pm\infty$:

$$\begin{aligned}
f(x) &= \frac{x+1}{x+2} \\
&= \frac{x+1}{x+2} \frac{1/x}{1/x} \\
f(x) &= \frac{1 + \frac{1}{x}}{1 + \frac{2}{x}}
\end{aligned}$$

1

$$\lim_{x \to \pm\infty} f(x) \;=\; \frac{1}{1} = 1$$

As x gets huge, the values $\frac{1}{x}$ and $\frac{2}{x}$ approach 0. We could abbreviate this conclusion as $f(\pm\infty) = 1$.

To draw the graph we use asymptotes; there's a horizontal asymptote at $y = 1$ and a vertical one at $x = -2$. We use dotted lines to sketch these asymptotes so that we won't confuse them with the sketch of our graph as in Figure 1.
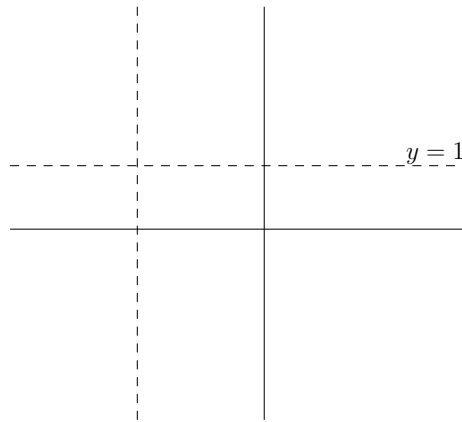

Figure 1: Axes and asymptotes.

The fact that $\lim_{x \to -2^+} f(x) = -\infty$ tells us that the graph plunges down as it approaches $x = -2$ from the right. As x approaches -2 from the left, the graph goes up toward positive infinity on the other side of the vertical asymptote. As x approaches negative infinity, the graph gets closer and closer to the horizontal asymptote $y = 1$, as it does when $x \to +\infty$.

We can almost finish this graph now, but there's one outstanding question. How do we know whether the graph crosses the line $y = 1$ or not?

Student: It can't dip below the line because there are no critical points.

That's exactly right. Because f' is never 0 the graph of f can't double back on itself, because the graph can't have any horizontal tangent lines.

We've practically got the entire graph now. We could improve it by adding details like the exact x- and y-intercepts, but we already understand the qualitative behavior of this function.

Let's look at the sign of the first derivative of the function to double check our graphing. To compute the derivative we can use the following algebraic trick:
$$f(x) = \frac{x+1}{x+2} = \frac{(x+2) - 1}{x+2} = 1 - \frac{1}{x+2} = 1 + (x+2)^{-1}.$$
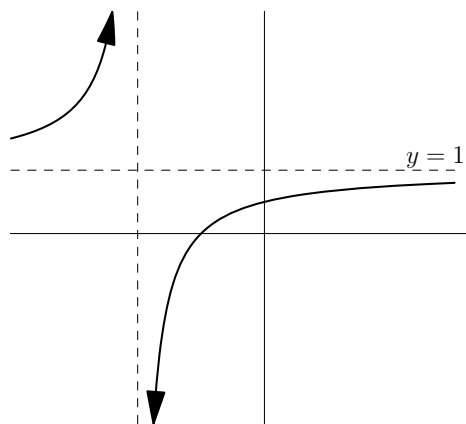
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Figure 2: Sketch of $y = \dfrac{x+1}{x+2}$.

So $f'(x) = 0 - (x+2)^{-2} = \frac{1}{(x+2)^2}$ for $x \neq -2$. (This calculation also shows us that the graph of $f(x)$ is a hyperbola similar to the graph of $\frac{1}{x}$.)

We can see from our graph that $f(x)$ is increasing from negative infinity to -2 and from -2 to positive infinity, so $f'(x)$ must be positive. The derivative we calculated is always positive, so we've successfully checked that result.

Despite the fact that we know that f is increasing on $(-\infty, 2)$ and $(2, \infty)$, we shouldn't think that f is increasing for all x. There's a break at $x = -2$, which is the most important feature of the function.

The second derivative of $f(x)$ is

$$f''(x) = \frac{-2}{(x+2)^3} \quad (x \neq -2).$$

This is negative when $(x+2)^3$ is positive, so $f''(x) < 0$ when $x > -2$ and the graph is concave down on the interval $(-2, \infty)$. The second derivative is positive when $(x+2)^3$ is negative, so the graph is concave up on the interval $(-\infty, -2)$. Again, this agrees with the graph we have drawn. It also confirms that the graph doesn't "wiggle".

Question: Are we defining "increasing" to mean that the first derivative is positive?

Answer: No. Whenever f' is positive the function is increasing, but it's possible for the function to be increasing even when f' is 0.

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