

JOEL LEWIS: Hi. Welcome back to recitation. In lecture, you've been doing related rates problems. I've got another example for you, here. So this one's a really tricky one. So I'm going to give you some time to work on it. But I really think you should try and work through this one yourself. It'll be well worth your effort. So we've got here--

OK, so this is a mouthful. We've got a 20-foot-long ladder and it's leaning against a 12-foot wall. But it's leaning over the wall. So 5 feet of the ladder project over the top of the wall. And then the bottom of the ladder is being pulled away from the wall at 5 feet per second. The question is, while this is going on, how quickly is the top of the ladder going downwards? How quickly is it approaching the ground?

So why don't you take a few minutes. Well, maybe more than a few for this one. It took me a while to work it out the first time. Take a few minutes, work this one out, come back, and we'll see how it went.

All right. So hopefully you've had some luck working out this problem on your own. Now let's work it out together.

So OK, so this is a sort of classic, really tricky related rates problem in that there's a lot of geometric work that you have to do at the beginning in order to get this right. And then once you get the geometry down, the calculus is basically totally straightforward. You know, you compute a couple of derivatives, you use the chain rule once, whatever. You know.

So OK, so let's start off by trying and drawing a careful picture. And then we'll have to figure out what the quantities that we're interested in are and the relationships between them.

So we have here, here's the ground. And we have a 12-foot wall. And we have a 20-foot ladder leaning against the wall and extending over it. So, my ladder. So here's my ladder. Here's my wall. All right. And this, the bottom of the ladder here, is what's getting pulled away from the wall at 5 units per second. 5 feet per second. So 5 feet per second is how fast that's going. And we're interested--

so while you do that, you know, the ladder is changing position. So the question is, how fast is the top of the ladder descending? So you know, it might also be moving in some other directions, but we're just interested in how quickly it's going straight down.

So, OK, so first, let's talk about the things that don't change. So the height of this wall is not changing. The wall is staying put the whole time. And the length of the ladder is not changing. That's staying put, as well. But what--

OK, so those are the fixed quantities in this problem. Basically, everything else is changing. So for example, this distance between the base of the wall and the base of the ladder, that's changing because the ladder is being pulled away from the wall. So we can give that a name. Let's call that-- that's a horizontal distance, so we can call that x , say. And the height of the top of the ladder is changing, so let's draw that in. That's what we're interested in. It's the height, so we can call it y .

And let's see. What else? Well OK, the whole ladder isn't changing in length, but the amount, but as this gets pulled, you know, this top point is sort of sliding down towards the wall, the amount of the ladder that extends over the wall is changing. So I guess we could choose. We could give this a variable name or we could give this part a variable name. I think I'm going to give the little one a variable name. I'll call it d for distance, I guess. And, OK. So if this is d , because the whole ladder has length 20, we have that this part, this segment has length 20 minus d .

OK so, I think that those quantities describe all the possible lengths of interest in this picture. So now, OK. So that's just the first, the set up. That's the first thing that has to happen. Now after we set up, we need to figure out what the relationships between these different variables are.

Well, let's see. What have we got? Well one thing we've got is that we've got a right triangle here. So the wall and the ground and the ladder. Those three segments form a right triangle. So OK. So we can apply the Pythagorean Theorem here. So that's one relationship we have, and that's going to give us a relationship between x and well, OK, 20 minus d . So that'll give us a relationship between x and d .

So this, so OK, so what does the Pythagorean Theorem say? So by the Pythagorean Theorem, we have x squared plus-- that's a 1-- 12 squared is equal to the quantity 20 minus d squared.

So that's one identity that we have, identity we have that relates these three quantities. And so that's good. Because we know how fast x is changing. And so with this identity, that means we can use related rates to figure out how fast d is changing, as well. So that's good.

That'll be a step in the right direction, but what we actually need to know is how fast y is changing. So-- right? Because that's what the question is asking for-- how quickly is the top of the ladder approaching the ground? So how, what's the rate of change in y with respect to time?

So I need another identity here in order to figure out what the relationship with y is and these other variables. And there are a couple of different ways to go about this. I think the one that I'm going to do-- so there is another right triangle here. So you could, if you wanted to name this segment, as well, then you could do another right triangle. But then you'd need a third relationship relating this segment to d or something like that.

So I'm not going to go that route, but I'm going to go do something related to that, which is that this bigger right triangle, right, which has the height of the top of the ladder from the ground and the whole length of the ladder as two of its sides, and a piece of the ground, so that bigger right triangle is similar to this smaller right triangle. Right? I mean, they have, they're right triangles and they have the same base angle there. So they're similar triangles.

So OK, so by similar triangles-- now I have to remember all my geometry, right? So the ratio of corresponding sides are equal, so in this case, the ratio of the hypotenuses is equal to the ratio of these legs. And that'll relate-- so the hypotenuses involve d . And the vertical legs, well, one of them is just constant and one of them involves y .

So that'll set up a relationship between d and y , and so then I'll have x linked to d and I'll have d linked to y . And so then we can use related rates to figure out what the relationship between y and x is and figure out the thing we're after, which is the rate of change in y .

So OK, so I haven't actually written anything down, yet. So what is the similar triangle? So here, let's look at the hypotenuses. So the big hypotenuse has length 20, and the small hypotenuse has length 20 minus d . Good. So that's the ratio of the hypotenuses, and so that has to be equal to the ratio of the corresponding legs. And so the big leg is y , and the small leg is 12. OK.

All right, so there's Pythagorean Theorem. There's similar triangles. So we now have the relationships that we're after, relating x and y and d all to each other. Great. So now we can do the calculus part of this problem, right? So, now we know--

oh, I guess, actually, we need a few more things. Take it back. So now we need the condition. We're after this at a particular moment in time, and at our particular moment in time-- so I haven't used this one given yet, that 5 feet of the wall-- oh, sorry-- 5 feet of the ladder projects over the wall. So OK, so, at our moment, so at the key moment, we have that d is equal to 5. OK, so that means 20 minus d is equal to 15 .

What else does that mean? So that means that at that moment, x in this right triangle is the third leg in a right triangle with hypotenuse 15 and one leg 12 . So that's one of your 3-4-5 triangles, but blown up a little bit. So we have at that moment, that x is equal to 9 . And OK, so at that moment, to figure out y , we can use this other relationship that we have. So 20 over 15 is equal to y over 12 , so y is equal to 20 over 15 times 12 . So that's 4 over 3 , so that's 16 .

So at our key moment in time, these are the values that we're going to end up plugging in. And also, dx/dt is equal to 5 . x is being increased. Increased? Yeah x is, right, so as the foot gets pulled away, x is getting bigger. So x is being increased at 5 units per second. All right, so these are all the things we're going to plug in. All right, good.

So now we are after dy/dt at this moment. And we don't have a relation-- so, we know dx/dt . We don't have a direct relationship between y and x , but we do have a direct relationship between y and d . So if we got dd/dt -- sorry-- dd/dt , then we could get dy/dt , and because of this relationship, we can get dd/dt .

All right. So let's start off by doing that. So OK, so this is an identity. It holds always, so we can differentiate it. And we're going to differentiate it with respect to t . x and d are functions of t . So OK. So differentiating this identity I get $2x$ times dx/dt plus-- well, OK, derivative of 144 with respect to t is 0 -- is equal to-- I guess I could expand this out, but it's easier just to use the chain rule right away. So this is equal to 2 times 20 minus d times the derivative of 20 minus d with respect to t , which is minus dd/dt . All right.

I beg the gods of math notation to forgive me for dd/dt . But, OK.

So, all right, so now-- this is always true. And what we want is the value of dd/dt at our particular moment in time. So at our particular moment in time we have-- well let's see, so OK, we have x , we have the dx/dt , and we have d . So we can just plug all those things in. So, at our moment--

let me see how I'm going to do this. I'll do it over here.

At our moment-- so $2x$ is 18 times dx/dt is 5 equals 2 times 20 minus d is 15 times minus dd/dt . And dd/dt is what we're after. So OK. So divide 2 into this is 9. 45 divided by 15 is 3, so dd/dt is equal to minus 3 at the moment that we're interested in. All right. Great.

So now we have dd/dt , and so now we can go back to the second relationship we have, the one that relates d and y , and we can do the same thing here. We can take a derivative, use the chain rule, use, and get a relationship between dy/dt and d and dd/dt . And then we'll be able to plug in all these values we have for our particular moment.

So OK, so I think I'm just going to differentiate this straight. There are some simplifications I could do first, but it'll work fine if we just do it straight. So OK, so I need to compute the derivative. This is an identity. I can take its derivative, so I differentiate the left-hand side with respect to t , so I get 20 over-- so this is a minus first power-- so I get minus 20 over 20 minus d quantity squared times-- I need the derivative of the bottom-- is minus dd/dt . And on the right-hand side I get dy/dt over 12.

So I have this relationship between d and y , I take its derivative, now I have a relationship that involves dy/dt , which is the thing that I'm after.

So, whew.

So now, I'm back at this plugging-in stage. So at our moment in time, I know what d is, and I just figured out what dd/dt is, so I can just plug those values in to figure out what dy/dt is. So OK, so let's do that. So at this moment in time we have-- so let me just multiply through by the 12 and that will let us solve-- so we have dy/dt is equal to 12 times-- well, so it's minus 20 over 15 squared, times minus minus 3. Times 3.

So this is-- all right. Arithmetic. Not my favorite thing in the world. So we've got two 3's here. We're going to end up with a 5 in the denominator, and then we've got 4 times minus 4, so that's minus 16 over 5. So.

All right. So that's the answer. Let's remember what we've done. So minus 16 over 5, that's a negative number. Why is it a negative number? Well, as we pull this, the bottom of this ladder here, away from the wall, the top of the ladder is going to fall downwards, it's going to get closer to the ground. And y is the vertical distance between the ground and the top of the ladder. So that's shrinking; y is shrinking and should have a negative derivative.

OK. Negative derivative, minus 16 over 5 feet per second. So it would take, if this instantaneous rate held up, which it won't in this situation-- if it held up it would take 5 seconds to get down, so it's not falling very quickly. OK. That seems about right.

Anything else? I guess we computed that dd/dt was also negative. Same thing here. When you pull this way, this length is going to shrink, so d is getting smaller, so it has negative derivative.

OK. So at this moment in time, the top of the ladder is falling-- the thing we're interested in-- at exactly 16 over 5 feet per second.

And we're all set.