Newton's Method: What Could Go Wrong?

Newton's method works (very) well if $|f'|$ is not too small, $|f''|$ is not too big, and x_0 starts near the solution x.

We're not going to discuss these conditions in detail, but let's see why they're there. If f'' is too large the graph would be sharply curved, in which case the tangent line might not be a good approximation to the graph and x_1 might not be close to the solution. There are a couple of things that can go wrong if x_0 is too far from x_1, which we'll discuss now.

If the error $E_0 = |x - x_0|$ is greater than 1 and $E_1 \sim E_0^2$, the error of your estimate could actually *increase* as you apply Newton's method.

In the example $f(x) = x^2 - 5$, if we had chosen $x_0 = -2$ we would have found the solution $-\sqrt{5}$ and not $\sqrt{5}$. This convergence to an unexpected root is illustrated in Fig. 1

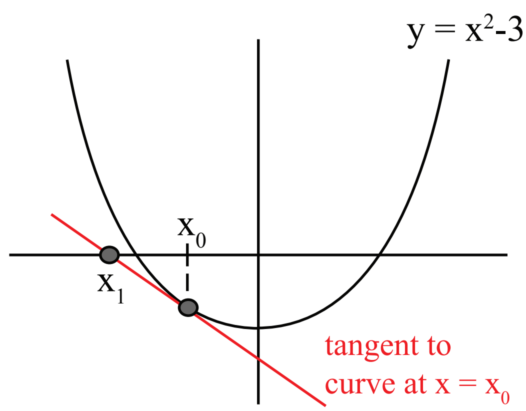

Figure 1: Newton's method converging to an unexpected root.

In the same example, if we chose $x_0 = 0$ then $f'(x_0) = 0$ and $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ is undefined.

Finally, there's a chance that Newton's method will cycle back and forth between two value and never converge at all. This failure is illustrated in Fig. 2; $x_2 = x_0$, $x_3 = x_1$, and so forth.

Newton's method is a good way of approximating solutions, but applying it requires some intelligence. You must beware of getting an unexpected result or no result at all. The better your initial guess at the solution, the more likely you are to get a correct result.

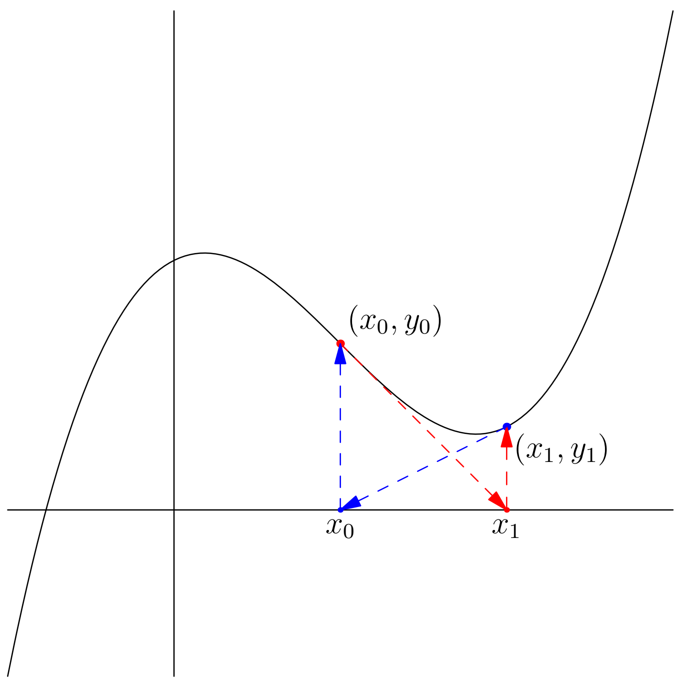

Figure 2: Newton's method cycling between x_0 and x_1.