$|\sin(b) - \sin(a)|$ vs. $|b - a|$

Use what we've learned about the mean value theorem to compare the values of $|\sin(b) - \sin(a)|$ and $|b - a|$.

Solution

We could solve this problem by starting with the mean value theorem, but we save some time if we recall from lecture that:

$$\min_{a \le x \le b} f'(x) \le \frac{f(b) - f(a)}{b - a} = f'(c) \le \max_{a \le x \le b} f'(x).$$

If $f(x) = \sin x$, then $f'(x) = \cos x$ and this inequality becomes:

$$\min_{a \le x \le b} \cos(x) \le \frac{\sin(b) - \sin(a)}{b - a} \le \max_{a \le x \le b} \cos(x).$$

We do not know the values of a and b, but we know that the cosine function ranges from -1 to 1. Replacing our unknown upper and lower bounds by these known bounds, we get:

$$-1 \le \frac{\sin(b) - \sin(a)}{b - a} \le 1,$$

or:

$$-(b - a) \le \sin(b) - \sin(a) \le b - a.$$

We conclude that $|\sin(b) - \sin(a)| \le |b - a|$. This is a surprisingly useful result derived solely from the fact that the slope of the sine curve is never greater than 1 or less than -1.