PROFESSOR: One correction from last time. Sorry to say, I forgot a very important factor when I was telling you what an average value is. If you don't put in that factor, it's only half off on the exam problem that will be given on this. So I would have gotten half off for missing out on this factor, too. So remember you have to divide by n here, certainly when you're integrating over 0 to n, the Riemann sum is the numerator here. And if I divide by n on that side, I've got to divide by n on the other side. This was meant to illustrate this idea that we're dividing by the total here. And we are going to be talking about average value in more detail. Not today, though. So this has to do with average value. And we'll discuss it in considerable detail in a couple of days, I guess.

Now, today I want to continue. I didn't have time to finish my discussion of the Fundamental Theorem of Calculus 2. And anyway it's very important to write it down on the board twice, because you want to see it at least twice. And many more times as well. So let's just remind you, the second version of the Fundamental Theorem of Calculus says the following. It says that the derivative of an integral gives you the function back again. So here's the theorem. And the way I'd like to use it today, I started this discussion last time. But we didn't get into it. And this is something that's on your problem set along with several other examples. Is that we can use this to solve differential equations. And in particular, for example, we can solve the equation y' = 1 / x with this formula. Namely, using an integral. L(x) is the integral from 1 to x of dt / t. The function f(t) is just 1 / t.

Now, that formula can be taken to be the starting place for the derivation of all the properties of the logarithm function. So what we're going to do right now is we're going to take this to be the definition of the logarithm. And if we do that, then I claim that we can read off the properties of the logarithm just about as easily as we could before. And so I'll illustrate that now. And there are a few other examples of this where somewhat more unfamiliar functions come up. This one is one that in theory we know something about.

The first property of this function is the one that's already given. Namely, its derivative is 1/x.

And we get a lot of information just out of the fact that its derivative is 1/x. The other thing that we need in order to nail down the function, besides its derivative, is one value of the function. Because it's really not specified by this equation, only specified up to a constant by this equation. But we nail down that constant when we evaluate it at this one place, L(1). And there we're getting the integral from 1 to 1 of dt / t, which is 0. And that's the case with all these definite integrals. If you evaluate them at their starting places, the value will be 0. And together these two properties specify this function L(x) uniquely.

Now, the next step is to try to think about what its properties are. And the first approach to that, and this is the approach that we always take, is to maybe graph the function, to get a feeling for it. And so I'm going to take the second derivative. Now, notice that when you have a function which is given as an integral, its first derivative is really easy to compute. And then its second derivative, well, you have to differentiate whatever you get. So it may or may not be easy. But anyway, it's a lot harder in the case when I start with a function to get to the second derivative. Here it's relatively easy. And these are the properties that I'm going to use. I won't really use very much more about it than that. And qualitatively, the conclusions that we can draw from this are, first of all, from this, for example we see that this thing is concave down every place. And then to get started with the graph, since I see I have a value here, which is L(1) = 0, I'm going to throw in the value of the slope. So L'(1), which I know is 1 over 1, that's reading off from this equation here, so that's 1.

And now I'm ready to sketch at least a part of the curve. So here's a sketch of the graph. Here's the point (1, 0), that is, x = 1, y = 0. And the tangent line, I know, has slope 1. And the curve is concave down. So it's going to look something like this. Incidentally, it's also increasing. And that's an important property, it's strictly increasing. That's because L'(x) is positive. And so, we can get from this the following important definition. Which, again, is working backwards from this definition. We can get to where we started with a log in our previous discussion. Namely, if I take the level here, which is y = 1, then that crosses the axis someplace. And this point is what we're going to define as e. So the definition of e is that it's the value such that L(e) = 1.

And again, the fact that there's exactly one such place just comes from the fact that this L' is positive, so that L is increasing. Now, there's just one other feature of this graph that I'm going to emphasize to you. There's one other thing which I'm not going to check, which you would ordinarily do with graphs. Once it's increasing there are no critical points, so the only other interesting thing is the ends. And it turns out that the limit as you go down to 0 is minus infinity. As you go over to the right here it's plus infinity. It does get arbitrarily high; it doesn't level off. But I'm not going to discuss that here. Instead, I'm going to just remark on one qualitative feature of the graph, which is this remark that the part which is to the left of 1 is below 0.

So I just want to remark, why is L(x) negative for x < 1. Maybe I don't have room for that, so I'll just put in here: x < 1. I want to give you two reasons. Again, we're only working from very first principles here. Just that-- the property that L' = 1/x, and L(1) = 0. So our first reason is that, well, I just said it. L(1) = 0. And L is increasing. And if you read that backwards, if it gets up to 0 here, it must have been negative before 0. So this is one way of seeing that L(x) is negative. There's a second way of seeing it, which is equally important. And it has to do with just manipulation of integrals. Here I'm going to start out with L(x), and its definition. Which is the integral from 1 to x, dt / t. And now I'm going to reverse the order of integration. This is the same, by our definition of our properties of integrals, as the integral from x to 1 with a minus sign dt / t.

Now, I can tell that this quantity is negative. And the reason that I can tell is that this chunk of it here, this piece of it, is a positive number. This part is positive. And this part is positive because x < 1. So the lower limit is less than the upper limit, and so this is interpreted - the thing in the green box is interpreted - as an area. It's an area. And so negative a positive quantity is negative, minus a positive quantity's negative. So both of these work perfectly well as interpretations. And it's just to illustrate what we can do.

Now, there's one more manipulation of integrals that gives us the fanciest property of the log. And that's the last one that I'm going to do. And you have a similar thing on your homework. So I'm going to prove that-- This is, as I say, the fanciest property of the log. On your homework, by the way, you're going to check that L(1/x) = -L(x). But we'll do this one. The idea is just to plug in the formula and see what it gives. On the left-hand side, I have 1 to ab, dt / t. That's L(ab). And then that's certainly equal to the left-hand side. And then I'm going to now split this into two pieces. Again, this is a property of integrals. That if you have an integral from one place to another, you can break it up into pieces. So I'm going to start at 1 but then go to a. And then I'm going to continue from a to ab. So this is the question that we have. We haven't proved this. Well, this one is actually true. If we want this to be true, we know by definition L(ab) is this. We know, we can see it, that L(a) is this. So the question that this boils down to is, we want to know that these two things are equal. We want to know that L(b) is that other integral there.

So let's check it. I'm going to rewrite the integral. It's the integral from-- sorry, from lower limit a to upper limit ab of dt / t. And now, again, to illustrate properties of integrals, the key property here that we're going to have to use is change of variables. This is a kind of a scaled integral where everything is multiplied by a factor of a from what we want to get to this L(b) quantity. And so this suggests that we write down t = au. That's going to be our trick. And if I use that new variable u, then the change in t, dt, is a du.

And as a result, I can write this as equal to an integral from, let's see, dt = a du. And t = au. So I've now substituted in for the integrand. But on top of this, with definite integrals, we also have to check the limits. And the limits work out as follows. When t = a, that's the lower limit. Let's just take a look. t = au. So that means that u is equal to, what? It's 1. Because a * 1 = a. So if t = a, this is if and only if. So this lower limit, which really in disguise was where t = a, becomes where u = 1. And similarly, when t = ab, u = b. So the upper limit here is b. And now, if you notice, we're just going to cancel these two factors here. And now we recognize that this is just the same as the definition of L(b). Because L(x) is over here in the box. And the fact that I use the letter t there is irrelevant; it works equally well with the letter u. So this is just L(b). Which is what we wanted to show. So that's an example, and you have one in your homework, which is a little similar.

Now, the last example, that I'm going to discuss of this type, I already mentioned last time. Which is the function F(x), which is the integral from 0 to x of e^(-t^2) dt. This one is even more exotic because unlike the logarithm it's a new function. It really is not any function that you can express in terms of the functions that we know already. And the approach, always, to these new functions is to think of what their properties are. And the way we think of functions in order to understand them is to maybe sketch them. And so I'm going to do exactly the same thing I did over here. So, what is it that I can get out of this? Well, immediately I can figure out what the derivative is. I read it off from the fundamental theorem. It's this. I also can figure out the value at the starting place. In this case, the starting place is 0. And the value is 0.

And I should check the second derivative, which is also not so difficult to compute. The second derivative is -2x e^(-x^2). And so now I can see that this function is increasing, because this derivative is positive, it's always increasing. And it's going to be concave down when x is positive and concave up when x is negative. Because there's a minus sign here, so the sign is negative. This is less than 0 when x is positive and greater than 0 when x is negative. And maybe to get started I'll remind you F(0) is 0. It's also true that F'(0)-- that just comes right out of this, F'(0) = e^(-0^2), which is 1. That means the tangent line again has slope 1. We do this a lot with functions. We normalize them so that the slopes of their tangent lines are 1 at convenient spots. So here's the tangent line of slope 1. We know this thing is concave down to the right and concave up to the left. And so it's going to look something like this. With an inflection point. Right?

Now, I want to say one more-- make one more remark about this function, or maybe two more remarks about this function, before we go on. Really, you want to know this graph as well as possible. And so there are just a couple more features. And one is enormously helpful because it cuts in half all of the work that you have. and that is the property that turns out that this function is odd. Namely, - F(-x) = -F(x). That's what's known as an odd function. Now, the reason why it's odd is that it's the antiderivative of something that's even. This function in here is even. And we nailed it down so that it was 0 at 0. Another way of interpreting that, and let me show it to you underneath, is the following. When we look at its derivative, its derivative, course, is the function e^x. Sorry, e^(-x^2). So that's this shape here. And you can see the slope is 0, but-- fairly close to 0, but positive along here. It's getting, this is its steepest point. This is the highest point here. And then it's leveling off again. The slope is going down, always positive. This is the graph of F' = e^(-x^2).

Now, the interpretation of the function that's up above is that the value here is the area from 0 to x. So this is area F(x). Maybe I'll color it in, decorate it a little bit. So this area here is F(x). Now, I want to show you this odd property, by using this symmetry. The graph here is even, so in other words, what's back here is exactly the same as what's forward. But now there's a reversal. Because we're keeping track of the area starting from 0 going forward. That's positive. If we go backwards, it's counted negatively. So if we went backwards to -x, we'd get exactly the same as that green patch over there. We'd get a red patch over here. But it would be counted negatively. And that's the property that it's odd. You can also check this by properties of integrals directly. That would be just like this process here. So it's completely analogous to checking this formula over there.

So that's one of the comments I wanted to make about this. And why does this save us a lot of time, if we know this is odd? Well, we know that the shape of this branch is exactly the reverse, or the reflection, if you like, of the shape of this one. What we want to do is flip it under the axis and then reflect it over that way. And that's the symmetry property of the graph of F(x). Now, the last property that I want to mention is what's happening with the ends. And at the end there's an asymptote, there's a limit here. So this is an asymptote. And the same thing down here, which will be exactly because of the odd feature, this'll be exactly negative. The opposite value over here. And you might ask yourself, what level is this, exactly. Now, that level turns out to be a very important quantity. It's interpreted down here as the area under this whole infinite stretch. It's all the way out to infinity.

So, let's see. What do you think it is? You're all clueless. Well, maybe not all of you, you're just afraid to say. So it's obvious. It's the square root of pi/2. That was right on the tip of your tongue, wasn't it?

STUDENT: Ah, yes.

PROFESSOR: Right, so this is actually very un-obvious, but it's a very important quantity. And it's an amazing fact that this thing approaches this number. And it's something that people worried about for many years before actually nailing down. And so what I just claimed here is that the limit as x approaches infinity of F(x) is equal to the square root of pi over 2. And similarly, if you do it to minus infinity, you'll get minus square root of pi over 2. And for this reason, people introduced a new function because they like the number 1. This function is erf, short for error function. And it's 2 over the square root of pi times the integral from 0 to x, e^(-t^2) dt. In other words, it's just our original, our previous function multiplied by 2 over the square root of pi. And that's the function which gets tabulated quite a lot. You'll see it on the internet everywhere, and it's a very important function. There are other normalizations that are used, and the discussions of the other normalizations are in your problems. This is one of them, and another one is in your exercises. The standard normal distribution. There are tons of functions like this, which are new functions that we can get at once we have the tool of integrals. And I'll write down just one or two more, just so that you'll see the variety.

Here's one which is called a Fresnel integral. On your problem set next week, we'll do the other Fresnel integral, we'll look at this one. These functions cannot be expressed in elementary terms. The one on your homework for this week was this one. This one comes up in Fourier analysis. And I'm going to just tell you maybe one more such function. There's a function which is called Li(x), logarithmic integral of x, which is this guy. The reciprocal of the logarithm, the natural log. And the significance of this one is that Li(x) is approximately equal to the number of primes less than x. And, in fact, if you can make this as precise as possible, you'll be famous for millennia, because this is known as the Riemann hypothesis. Exactly how closely this approximation occurs. But it's a hard problem, and already a century ago the prime number theorem, which established this connection was extremely important to progress in math. Yeah, question.

STUDENT: [INAUDIBLE]

PROFESSOR: Is this stuff you're supposed to understand. That's a good question. I love that question. The answer is, this is, so we launched off into something here. And let me just explain it to you. I'm going to be talking a fair amount more about this particular function, because it's associated to the normal distribution. And I'm going to let you get familiar with it. What I'm doing here is purely cultural. Well, after this panel, what I'm doing is purely cultural. Just saying there's a lot of other beasts out there in the world. And one of them is called C of x-- So we'll have a just a very passing familiarity with one or two of these functions. But there are literally dozens and dozens of them. The only thing that you'll need to do with such functions is things like understanding the derivative, the second derivative, and tracking what the function does. Sketching the same way you did with any other tool. So we're going to do this type of thing with these functions. And I'll have to lead you through. If I wanted to ask you a question about one of these functions, I have to tell you exactly what I'm aiming for. Yeah, another question.

STUDENT: [INAUDIBLE]

PROFESSOR: Yeah, I did. I called these guys Fresnel integrals. The guy's name is Fresnel. It's just named after a person. But, and this one, Li is logarithmic integral, it's not named after a person. Logarithm is not somebody's name. So look, in fact this will be mentioned also on a problem set, but I don't expect you to remember these names. In particular, that you definitely don't want to try to remember. Yes, another question.

STUDENT: [INAUDIBLE]

PROFESSOR: The question is, will we prove this limit. And the answer is yes, if we have time. It'll be in about a week or so. We're not going to do it now. It takes us quite a bit of work to do it. OK.

I'm going to change gears now, I'm going to shift gears. And we're going to go back to a more standard thing which has to do with just setting up integrals. And this has to do with understanding where integrals play a role, and they play a role in cumulative sums, in evaluating things. This is much more closely associated with the first Fundamental Theorem. That is, we'll take, today we were talking about how integrals are formulas for functions. Or solutions to differential equations. We're going to go back and talk about integrals as being the answer to a question as opposed to what we've done now. So in other words, and the first example, or most of the examples for now, are going to be taken from geometry. Later on we'll get to probability.

And the first topic is just areas between curves. Here's the idea. If you have a couple of curves that look like this and maybe like this, and you want to start at a place a and you want to end at a place b, then you can chop it up the same way we did with Riemann sums. And take a chunk that looks like this. And I'm going to write the thickness of that chunk. Well, let's give these things names. Let's say the top curve is f(x), and the bottom curve is g(x). And then this thickness is going to be dx. That's the thickness. And what is the height? Well, the height is the difference between the top value and the bottom value. So here we have (f(x) - g(x)) dx. This is, if you like, base times-- Whoops, backwards. This is height, and this is the base of the rectangle. And these are approximately correct. But of course, only in limit when this is an infinitesimal, is it exactly right.

In order to get the whole area, I have add these guys up. So I'm going to integrate from a to b. That's summing them, that's adding them up. And that's going to be my area. So that's the story here. Now, let me just say two things about this. First of all, on a very abstract level before we get started with details of more complicated problems. The first one is that every problem that I'm going to be talking about from now on for several days, involves the following collection of-- the following goals. I want to identify something to integrate. That's called an integrand. And I want to identify what are known as the limits. The whole game is simply to figure out what a, b, and this quantity is here. Whatever it is.

And the minute we have that, we can calculate the integral if we like. We have numerical procedures or maybe we have analytic procedures, but anyway we can get at the integral. The goal here is to set them up. And in order to set them up, you must know these three things. The lower limit, the upper limit, and what we're integrating. If you leave one of these out, it's like the following thing. I ask you what the area of this region is. If I left out this end, how could I possibly know? I don't even know where it starts, so how can I figure out what this area is if I haven't identified what the left side is. I can't leave out the bottom. It's sitting here, in this formula. Because I need to know where it is.

And I need to know the top and I need to know this side. Those are the four sides of the figure. If I don't incorporate them into the information, I'll never get anything out. So I need to know everything. And I need to know exactly those things, in order to have a formula for the area.

Now, when this gets carried out in practice, as we will do now in our first example, it's more complicated than it looks. So here's our first example: Find the area between $x = y^2$ and $y = x - 2$. This is our first example. Let me make sure that I chose the example that I wanted to. Yeah. Now, there's a first step in figuring these things out. And this is that you must draw a picture. If you don't draw a picture you'll never figure out what this area is, because you'll never figure out what's what between these curves. The first curve, $y = x^2$, is a parabola. But x is a function of y. It's pointing this way. So it's this parabola here. That's $y = x^2$. Whoops, $x = y^2$.

The second curve is a line, a straight line of slope 1, starting at $x = 2$, $y = 0$. It goes through this place here, which is 2 over and has slope 1, so it does this. And this shape in here is what we mean by the area between the curves. Now that we see what it is, we have a better idea of what our goal is. If you haven't drawn it, you have no hope.

Now, I'm going to describe two ways of getting at this area here. And the first one is motivated by the shape that I just described right here. Namely, I'm going to use it in a straightforward way. I'm going to chop things up into these vertical pieces just as I did right there. Now, here's the difficulty with that. The difficulty is that the upper curve here has one formula but the lower curve shifts from being a part of the parabola to being a part of the straight line. That means that there are two different formulas for the lower function. And the only way to accommodate that is to separate this up into two halves. Separate it out into two halves. I'm going to have to divide it right here. So we must break it into two pieces and find the integral of one half and the other half. Question?

STUDENT: [INAUDIBLE]

PROFESSOR: So, you're one step ahead of me. We'll also have to be sure to distinguish between the top branch and the bottom branch of the parabola, which we're about to do. Now, in order to distinguish what's going on I actually have to use multi colors here. And so we will do that. First there's the top part, which is orange. That's the top part. I'll call it top. And then there's the bottom part, which has two halves. They are pink, and I guess this is blue. All right, so now let's see what's happening. The most important two points that I have to figure out in order to get started here. Well, really I'm going to have to figure out three points, I claim. I'm going to have to figure out where this point is. Where this point is, and where that point is. If I know where these three points are, then I have a chance of knowing where to start, where to end, and so forth. Another question.

STUDENT: [INAUDIBLE]

PROFESSOR: Could you speak up?

STUDENT: [INAUDIBLE]

PROFESSOR: The question is, why do we need to split up the area. And I think in order to answer that question further, I'm going to have to go into the details of the method, and then you'll see where it's necessary. So the first step is that I'm going to figure out what these three points are. This one is kind of easy; it's the point (0, 0). This point down here and this point up here are intersections of the two curves. I can identify them by the following equation. I need to see where these curves intersect. At what, well, if I plug in $x = y^2$, I get $y = y^2 - 2$. And then I can solve this quadratic equation. $y^2 - y - 2 = 0$. So $(y - 2) (y+1) = 0$. $= 0$. And this is telling me that $y = 2$ or $y = -1$.

So I've found $y = -1$. That means this point down here has second entry -1. Its first entry, its x-value, I can get from this formula here or the other formula. I have to square, this, $-1^2 = 1$. So that's the formula for this point. And the other point has second entry 2. And, again, with his formula $y = x^2$, I have to square y to get x, so this is 4. Now, I claim I have enough data to get started. But maybe I'll identify one more thing. I need the top, the bottom left, and the bottom right. The top is the formula for this branch of $x = y^2$, which is in the positive y region. And that is y is equal to square root of x. The bottom curve, part of the parabola, so this is the bottom left, is y equals minus square root x. That's the other branch of the square root. And this is exactly what you were asking before. And this is, we have to distinguish between these two. And the point is, these formulas really are different. They're not the same.

Now, the last bit is the bottom right chunk here, which is this pink part. Bottom right. And that one is the formula for the line. And that's $y = x - 2$. Now I'm ready to find the area. It's going to be in two chunks. This is the left part, plus the right part. And the left part, and I want to set it up as an integral, I want there to be a dx and here I want to set up an integral and I want it to be dx. I need to figure out what the range of x is. So, first I'm going to-- well, let's leave ourselves a little more room than that. Just to be safe. OK, here's the right. So here we have our dx. Now, I need to figure out the starting place and the ending place. So the starting place is the leftmost place. The leftmost place is over here. And $x = 0$ there. So we're going to travel from this vertical line to the green line. Over here. And that's from 0 to 1. And the difference between the orange curve and the blue curve is what I call top and bottom left, over there. So that is square root of x minus minus square root of x. Again, this is what I call top, and this was bottom. But only the left.

I claim that's giving me the left half of this, the left section of this diagram. Now I'm going to do the right section of the diagram. I start at 1. The lower limit is 1. And I go all the way to this point here. Which is the last bit. And that's going to be x = 4. The upper limit here is 4. And now I have to take the difference between the top and the bottom again. The top is square root of x all over again. But the bottom has changed. The bottom is now the quantity x - 2. Please don't forget your parenthesis. There's going to be minus signs and cancellations.

Now, this is almost the end of the problem. The rest of it is routine. We would just have to evaluate these integrals. And, fortunately, I'm going to spare you that. We're not going to bother to do it. That's the easy part. We're not going to do it. But I'm going to show you that there's a much quicker way with this integral. And with this area calculation. Right now. The quicker way is what you see when you see how long this is. And you see that there's another device that you can use that looks similar in principle to this, but reverses the roles of x and y. And the other device, which I'll draw over here, schematically. No, maybe I'll draw it on this blackboard here. So, Method 2, if you like, this was Method 1, and we should call it the hard way. Method 2, which is better in this case, is to use horizontal slices. Let me draw the picture, at least schematically.

Here's our picture that we had before. And now instead of slicing it vertically, I'm going to slice it horizontally. Like this. Now, the dimensions have different names. But the principle is similar. The width, we now call dy. Because it's the change in y. And this distance here, from the left end to the right end, we have to figure out what the formulas for those things are. So on the left, maybe I'll draw them color coded again. So here's a left. And, whoops, orange is right, I guess. So here we go. So we have the left - which is this green - is x = y^2 And the right, which is orange, is y = x - 2. And now in order to use this, it's going to turn out that we want to write x as-- we want to reverse roles. So we want to write this as x is a function of y. So we'll use it in this form.

And now I want to set up the integral for you. This time, the area is equal to an integral in the dy variable. And its starting place is down here. And its ending place is up there. This is the lowest value of y, and this is the top value of y. And we've already computed those things. The lowest level of y is -1. So this is $y = -1$. And this top value is $y = 2$. So this goes from -1 to 2. And now the difference is this distance here, the distance between the rightmost point and the leftmost point. Those are the two dimensions. So again, it's a rectangle but its horizontal is long and its vertical is very short. And what are they? It's the difference between the right and the left. The right-hand is $y + 2$, and the right-hand is y^2. So this is the formula.

STUDENT: [INAUDIBLE]

PROFESSOR: What was the question? Why is it right minus left? That's very important. Why is it right minus left? And that's actually the point that I was about to make. Which is this. That $y + 2$, which is the right, is bigger than y^2, which is the left. So that means that $y + 2 - y^2$ is positive. If you do it backwards, you'll always get a negative number and you'll always get the wrong answer. So this is the right-hand end minus the left-hand end gives you a positive number. And it's not obvious, actually, where you are. There's another double-check, by the way. When you look at this quantity, you see that the ends pinch. And that's exactly the crossover points. That is, when $y = -1$, $y + 2 - y^2 = 0$. And when $y = 2$, $y + 2 - y^2 = 0$. And that's not an accident, that's exactly the geometry of the shape that we picked out there. So this is the technique. Now, this is a much more routine integral. I'm not going to carry it out, I'll just do one last step. Which is that this is $y^2 / 2 + 2y - y^3 / 3$, evaluated at -1 and 2. Which, if you work it out, is 9/2. So we're done for today. And tomorrow we'll do more volumes, more things, including three dimensions.