Example: Find the area between $x = y^2$ and $y = x - 2$

First, graph these functions. If skip this step you'll have a hard time figuring out what the boundaries of your area is, which makes it very difficult to compute the area!
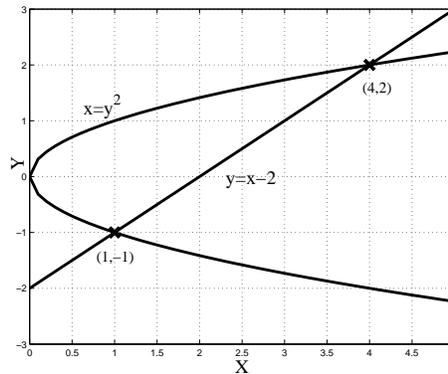
Figure 1: Finding the area between two intersecting graphs.

There are two ways of finding the area between these two curves: the hard way and the easy way.

Hard Way: slice it vertically

First, we'll try chopping the region up into vertical rectangles.
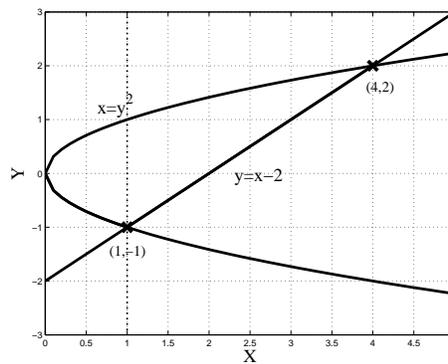
Figure 2: The area between $x = y^2$ and $y = x - 2$ split into two subregions.

If we slice the region between the two curves this way, we need to consider two different regions. Where $x > 1$, the region's lower bound is the straight line. For $x < 1$, however, the region's lower bound is the lower half of the sideways parabola. The left end of the region is at $x = 0$.

We must find the crossing points of the two curves; in other words, we find the values of x and y that satisfy both equations simultaneously.

$$x = y^2 \quad \text{and} \quad y = x - 2$$

so:

$$
\begin{aligned}
y - 2 &= y^2 \\
y^2 - y - 2 &= 0 \\
(y - 2)(y + 1) &= 0
\end{aligned}
$$

We conclude that:

$$y = 2 \quad \text{or} \quad y = -1.$$

We can plug these values of y back in to either equation to find the associated x values:

$$
\begin{aligned}
y &= x - 2 \\
2 &= x - 2 \\
4 &= x.
\end{aligned}
$$

If we perform a similar equation with $y = -1$ we'll find that the two points of intersection are $(1, -1)$ and $(4, 2)$.

The equation of the upper half of the sideways parabola is $y = \sqrt{x}$ and that of the lower half is $y = -\sqrt{x}$. The equation of the lower right hand boundary of the region is just $y = x - 2$.

We find the area A between the two curves by integrating the difference between the top curve and the bottom curve in each region:

$$A = \underbrace{\int_0^1 (\overbrace{\sqrt{x}}^{\text{top}} - \overbrace{(-\sqrt{x})}^{\text{bottom-l}})) \, dx}_{\text{left}} + \underbrace{\int_1^4 (\overbrace{\sqrt{x}}^{\text{top}} - \overbrace{(x - 2)}^{\text{bottom-r}}) \, dx}_{\text{right}}$$

The rest of this calculation is easy; just evaluate the integrals.

$$
\begin{aligned}
A &= 2 \int_0^1 \sqrt{x} \, dx + \int_1^4 (-x + \sqrt{x} + 2) \, dx \\
&= 2 \left[\frac{2}{3} x^{3/2} \right]_0^1 + \left[-\frac{1}{2} x^2 + \frac{2}{3} x^{3/2} + 2x \right]_1^4 \\
&= 2 \left(\frac{2}{3} - 0 \right) + \left(-\frac{4^2}{2} + \frac{2}{3} \cdot 4^{3/2} + 8 \right) - \left(-\frac{1}{2} + \frac{2}{3} + 2 \right) \\
&= \frac{4}{3} - 8 + \frac{16}{3} + 8 + \frac{1}{2} - \frac{2}{3} - 2 \\
A &= \frac{9}{2}.
\end{aligned}
$$

Easy Way: Slice it horizontally

There's a much quicker way to complete this area calculation; you should look for an easier way as soon as you notice the need to split the region into parts. The quicker way is similar in principle but reverses the roles of x and y; in this method we slice the area in question into horizontal rectangles.
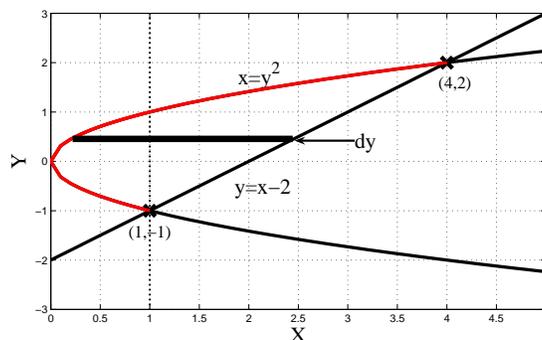
Figure 3: The area between $x = y^2$ and $y = x - 2$ and one horizontal rectangle.

The height of these rectangles is dy; we get their width by subtracting the x-coordinate of the edge on the left curve from the x-coordinate of the edge on the right curve. (If you get mixed up and subtract the right from the left you'll get a negative answer.) The left curve is the sideways parabola $x = y^2$. The right curve is the straight line $y = x - 2$ or $x = y + 2$.

The limits of integration come from the points of intersection we've already calculated. In this case we'll be adding the areas of rectangles going from the bottom to the top (rather than left to right), so from $y = -1$ to $y = 2$.

$$
\begin{aligned}
A &= \int_{y=-1}^{y=2} \left[(y+2) - y^2 \right] dy \\
&= \left[\frac{-y^3}{3} + 2y + \frac{y^2}{2} \right]_{-1}^{2} \\
&= \left(\frac{4}{2} + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) \\
A &= \frac{9}{2}
\end{aligned}
$$

You'll notice that if you plug the limits of integration into the integrand, you get 0. This makes sense; as y goes toward -1 and 2 the width of the rectangles approaches 0.

3

18.01SC Single Variable Calculus
Fall 2010