

## Repeated Factors

Now that we've seen the basic method of partial fractions we need to address possible complications. The first complication we'll consider is what to do if the factors in the denominator are not distinct — i.e. if some of the factors are repeated. In order for this technique to work, the degree of the numerator must still be less than the degree of the denominator.

Example:  $\frac{x^2 + 2}{(x - 1)^2(x + 2)}$

Again, step 1 has already been done for us. In the set-up, step 2, we need to add a second term for the second factor of  $(x - 1)$ .

$$\frac{x^2 + 2}{(x - 1)^2(x + 2)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x + 2}$$

In general, when you have  $(x - a)^n$  in the denominator you get  $n$  corresponding terms in your sum; one for each of the powers  $(x - a)^1, (x - a)^2, \dots, (x - a)^n$ . If the expression were:

$$\frac{x^2 + 2}{(x - 1)^3(x + 2)},$$

the setup would need to include another term with  $(x - 1)^3$  in the denominator.

**Question:** Why does it have to be squared?

**Answer:** This is a good question; we'll use an analogy to hint at the answer. The reasoning behind the  $(x - 1)^2$  is similar to the reasoning behind place value in the decimal expansion of a number. Similarly, we could expand the fraction  $\frac{7}{16}$  as:

$$\frac{7}{16} = \frac{0}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}.$$

Because of the  $2^4$  in the denominator of  $\frac{7}{16}$  we need to use powers of 2 up to  $2^4$  in the denominator to represent  $\frac{7}{16}$  in this way.

When you have repeated factors in the denominator the cover-up method still works, but it doesn't work quite as well. The cover-up method will work for the coefficients  $B$  and  $C$  but *not* for  $A$ .

We start by using the cover-up method to solve for  $C$ :

$$\begin{aligned} \frac{x^2 + 2}{(x - 1)^2(x + 2)} &= \frac{\cancel{A}}{\cancel{x - 1}} + \frac{\cancel{B}}{(x - 1)^2} + \frac{C}{\cancel{x + 2}} \\ \frac{(-2)^2 + 2}{((-2) - 1)^2} &= C \\ \frac{6}{9} &= C \\ C &= \frac{2}{3} \end{aligned}$$

The cover-up method will also work to find  $B$ :

$$\begin{aligned} \frac{x^2 + 2}{\cancel{(x-1)^2}(x+2)} &= \frac{\cancel{A}}{\cancel{x-1}} + \frac{B}{\cancel{(x-1)^2}} + \frac{\cancel{C}}{\cancel{x+2}} \\ \frac{1^2 + 2}{1 + 2} &= B \\ B &= 1 \end{aligned}$$

Let's do that again the slow way to see why it worked:

$$\begin{aligned} (x-1)^2 \frac{x^2 + 2}{(x-1)^2(x+2)} &= \frac{A}{x-1}(x-1)^2 + \frac{B}{(x-1)^2}(x-1)^2 + \frac{C}{x+2}(x-1)^2 \\ \frac{x^2 + 2}{(x+2)} &= A(x-1) + B + \frac{C}{x+2}(x-1)^2 \end{aligned}$$

When we set  $x = 1$ , every term with a multiple of  $(x - 1)$  in it becomes zero and we're left with the value of  $B$ .

We can't get everything to cancel so nicely to give us the value of  $A$ , which is divided only by a single power of  $(x - 1)$ . If we multiply through by just  $(x - 1)$  we'll still have an  $(x - 1)$  in the denominator of the  $B$  term which would cause a division by 0. If we multiply through by  $(x - 1)^2$  the  $A$  term cancels completely.

So we have to find another strategy to solve for  $A$ . Let's try plugging in Professor Jerison's favorite number,  $x = 0$ . (Unfortunately, if you use  $x = 0$  in solving for other terms in the decomposition you can't use it here. So far we've used  $x = 1$  and  $x = -2$  so it's ok to use  $x = 0$ .) Plugging in  $x = 0$ ,  $B = 1$  and  $C = \frac{2}{3}$ , we get:

$$\begin{aligned} \frac{0^2 + 2}{(0-1)^2(0+2)} &= \frac{A}{0-1} + \frac{1}{(0-1)^2} + \frac{2/3}{0+2} \\ \frac{2}{2} &= \frac{A}{-1} + \frac{1}{1} + \frac{2/3}{2} \\ 1 &= -A + 1 + \frac{1}{3} \\ A &= \frac{1}{3} \end{aligned}$$

This is a lot of algebra, but if we're careful and thorough we get the right answer and our rational expression becomes easy to integrate.

**Question:** If  $x = 0$  has already been used, what should we do?

**Answer:** Pick something else, like  $x = 1$ .

**Question:** If you had more powers would you have more variables?

**Answer:** Yes. As the degree of the denominator goes up, the number of variables goes up.

**Question:** How would you solve it if you had two unknowns?

**Answer:** When we plug in  $x = 0$  (or whatever) we'll get an equation in however many unknowns are left. There are methods of solving systems of equations for those variables which we'll learn more about later.

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